Data-Driven Algorithms for Transport

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Outline

We can use data to approximate operators and improve calculations.

- In scientific computing we are used to taking a known operator and making approximations to it.
- It is possible to use the action of an operator and use just the action of the operator to generate approximations to it.
  - This is the basis for many Krylov methods.
- In this talk I will detail how we can use the action of radiation transport operators to
  - Estimate time eigenvalues present in a subcritical system, and
  - Compute the slowly converging modes in source iteration to accelerate convergence without the need for diffusion-based preconditioning.
- I will also talk about how we can generate a low-rank approximation to the time-dependent transport system.

Outline

We begin with a sequence of vectors related by an operator.

- Consider a sequence of vectors $\{y_0, y_1, \ldots, y_K\}$ where $y_k \in \mathbb{R}^N$.
- The vectors are related by a potentially unknown linear operator of size $N \times N$, A, as
$$y_{k+1} = Ay_k.$$
- If we construct the $N \times K$ *data matrices* Y_+ and Y_-,
$$Y_+ = \begin{pmatrix} | & | & & | \\ y_1 & y_2 & \cdots & y_K \\ | & | & & | \end{pmatrix} \qquad Y_- = \begin{pmatrix} | & | & & | \\ y_0 & y_1 & \cdots & y_{K-1} \\ | & | & & | \end{pmatrix}$$
we can write
$$Y_+ = AY_-.$$
- At this point we only need to know the data vectors y_k, they could come from a calculation, measurement, etc.
- As $K \to \infty$ we could hope to infer properties about A.

The SVD gives a representation of the data matrices.

- We take the thin singular value decomposition (SVD) of Y_- to write

$$Y_- = U\Sigma V^{\mathrm{T}},$$

  where U is a $N \times K$ orthogonal matrix, Σ is a diagonal $K \times K$ matrix with non-negative entries on the diagonal, and V is a $K \times K$ orthogonal matrix.
- The SVD requires $O(NK^2)$ operations to compute.
- Later, we will want $K \ll N$, if, for example, N is the number of unknowns in a transport calculation.
- Also, if the column rank of $Y_- < K$, then there is a further reduction in the SVD size.
- The matrix U has columns that forms an orthonormal basis for the row space of $Y_- \subset \mathbb{R}^N$.
- Using the SVD we get

$$Y_+ = AU\Sigma V^{\mathrm{T}}.$$

- If there are only $r < K$ non-zero singular values in Σ, we use the compact SVD where U is $N \times r$, Σ is $r \times r$, and V is $K \times K$.

DMD gives a low rank approximation to the operator.

- We can rearrange the relationship between Y_+ and Y_- to be

$$Y_+ = AU\Sigma V^{\mathrm{T}} \qquad \rightarrow \qquad U^{\mathrm{T}} AU = U^{\mathrm{T}} Y_+ V\Sigma^{-1}.$$

- Define $\tilde{A} = U^{\mathrm{T}} AU = U^{\mathrm{T}} Y_+ V\Sigma^{-1}$. This is a rank K approximation to A.
- Using the approximate operator $\tilde{A}$, we can now find out information about A.
- The eigenvalues/vectors of $\tilde{A}$,

$$\tilde{A}w = \lambda w,$$

  are used to define the dynamic modes of A:

$$\varphi = \frac{1}{\lambda} U^{\mathrm{T}} Y_+ V\Sigma^{-1} w.$$

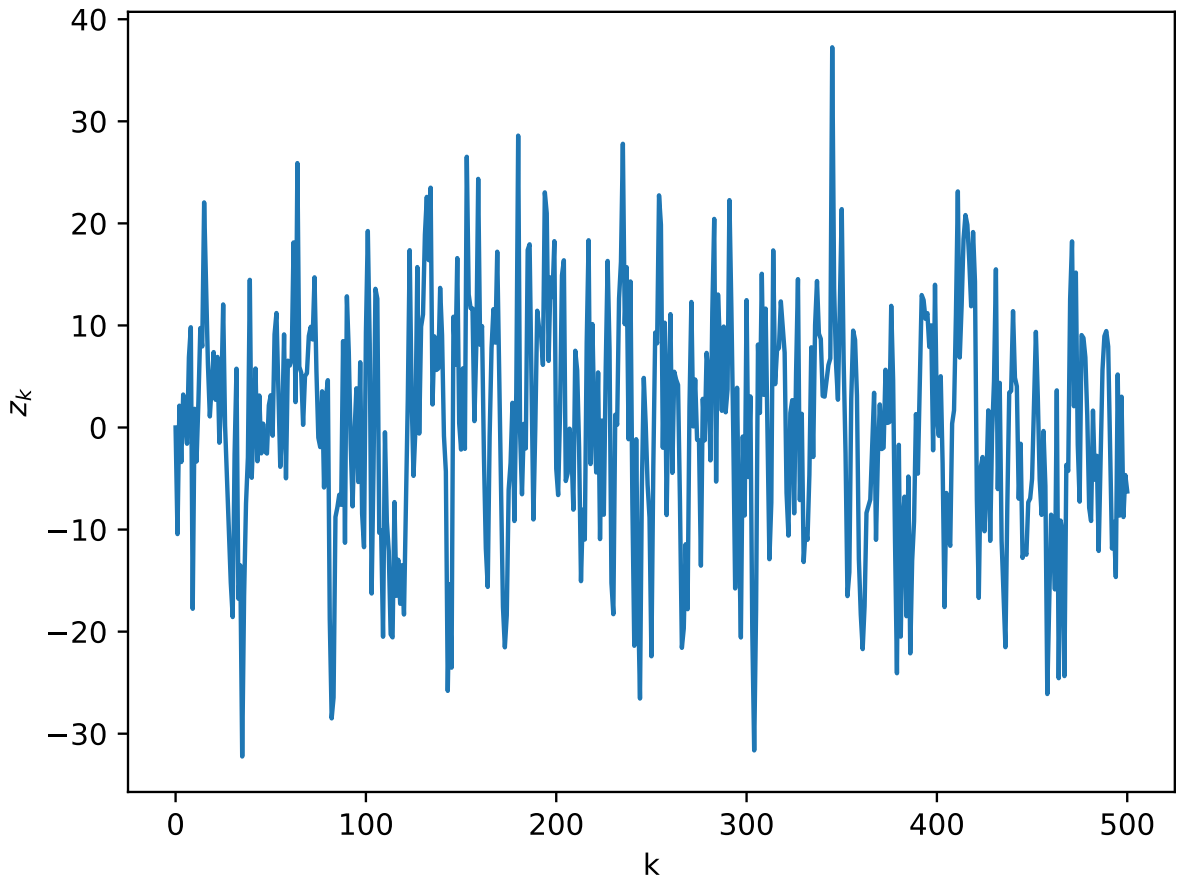
- The dynamic mode decomposition (DMD) of the data matrix Y_+ is then the decomposition of into vectors φ. The mode with the largest norm of λ is said to be the dominant mode.

Example of operator approximation demonstrates the efficacy.

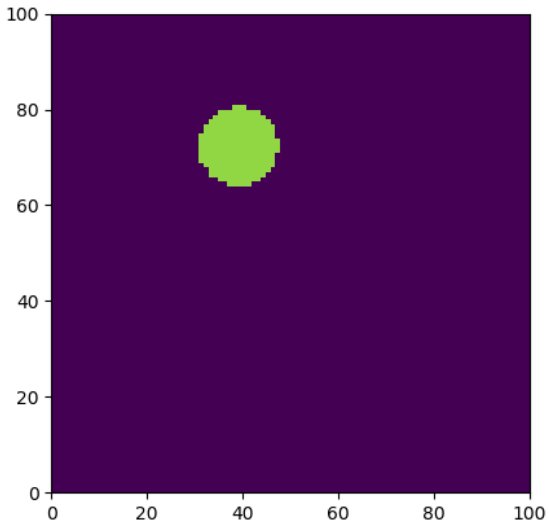
- Consider the sequence

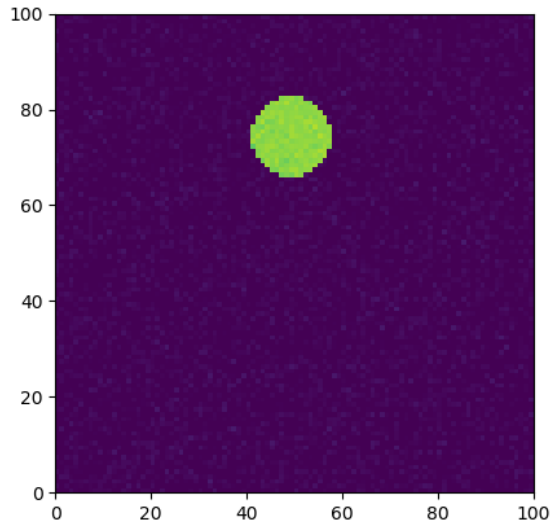
$$z_{k+1} = az_k + n_k,$$

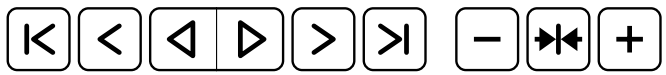
  where $a = 0.5$, and $n_k \sim \mathcal{N}(0, 10^2)$.
- Using $K = 500$, we estimate $a = 0.506552$ from the data below.



Evolution without an operator is possible: DMD infers the operator from the data.



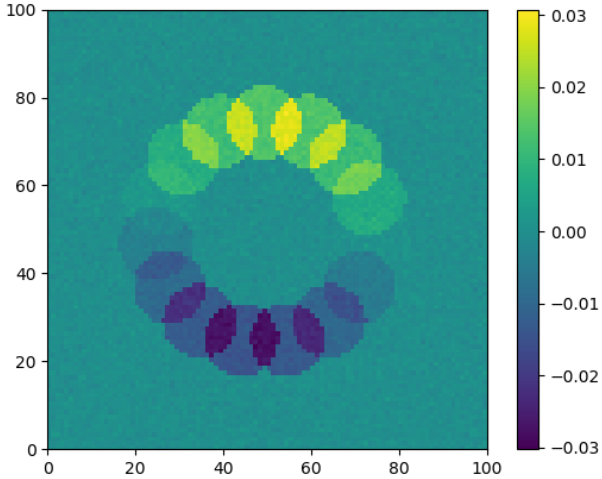




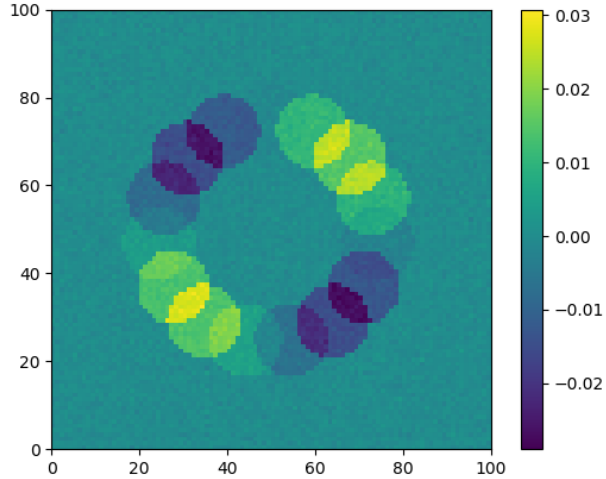
Left: Data generated by moving a circle in a periodic motion with added noise. The data has two periods of motion.

Right: Reconstruction generated by approximating $\tilde{A}$ using one period of frames and starting from frame 1.

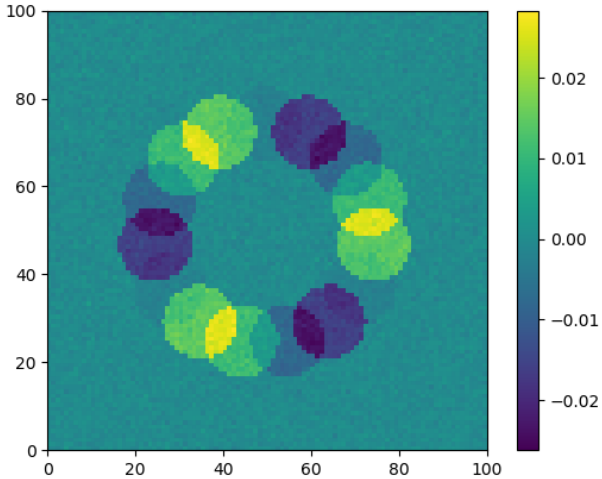
The DMD modes of the inferred operator are what we would expect.



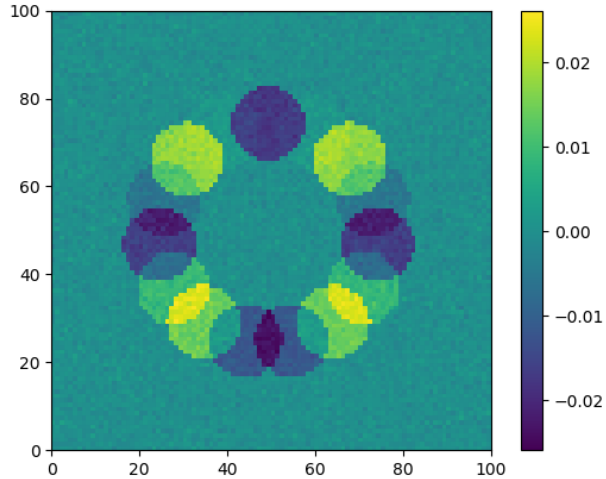
Dominant DMD mode: $U\varphi_1$



Second DMD mode: $U\varphi_2$



Third DMD mode: $U\varphi_3$



Fourth DMD mode: $U\varphi_4$

Outline

Current methods of estimating time-eigenvalues have issues for subcritical systems.

- In neutron transport for time-dependent problems alpha eigenvalues (also called time eigenvalues) are important quantities to understand system dynamics and safety.
  - These eigenvalues characterize the system evolution in terms of functions of the form $Ce^{\alpha t}$.
- The more well-known k-eigenvalue gives information about the long-term behavior of the system, but is less useful for diagnosing many experiments.
- Additionally, most alpha eigenvalue solvers have issues with subcritical problems due to "negative absorption".
- Furthermore, for subcritical systems it has been shown that the rightmost eigenvalue in the complex plane is not necessarily meaningful.
  - There can be an eigenvalue with negative real part that is arbitrarily close to 0 and decays arbitrarily slowly.
  - These eigenvalues correspond to the time scale of slow moving neutrons crossing the system.

DMD can be used to estimate eigenvalues of the matrix exponential.

- If we consider a sequence of vectors that are solutions to the system of differential equations,
$$\frac{\partial y}{\partial t} = Ay(t),$$
and are separated by a time, Δt, the relationship between vectors is
$$y^{n+1} = e^{A\Delta t} y^n.$$
- As before we can define Y_- and Y_+, compute the SVD of $Y_- = USV^*$, and approximate the matrix exponential:
$$U^{\mathrm{T}} e^{A\Delta t} U = U^{\mathrm{T}} Y_+ V\Sigma^{-1}.$$
- One can show the following:
  - The eigenvalues of $U^{\mathrm{T}} e^{A\Delta t} U$ are also eigenvalues of $e^{A\Delta t}$.
  - If α is an eigenvalue of A, then $e^{\alpha\Delta t}$ is an eigenvalue of $e^{A\Delta t}$.
  - The eigenvectors of A are the same as those from $e^{A\Delta t}$.

We can cast the alpha eigenvalue problem into this framework.

- Consider the time-dependent transport equation

$$\frac{\partial \psi}{\partial t} = A\psi,$$

where the discretized transport operator A is given by

$$A = v(E)(-\Omega \cdot \nabla + -\sigma_{\mathrm{t}} + \mathcal{S} + \mathcal{F}),$$

with $\mathcal{S}$ and $\mathcal{F}$ the scattering and fission operators.

- The alpha eigenvalues of the transport equation satisfy the following relationship

$$A\psi = \alpha\psi,$$

that is, we are interested in the eigenvalues of the complete transport operator.

- The importance of the alpha eigenvalue can be found by noticing that if $\psi(t) = \psi_0 e^{\alpha t}$, then α is an eigenvalue of the transport operator.
  - $\alpha > 0$ implies that the system is supercritical and will have a divergent number of neutrons as $t \to \infty$.
  - $\alpha = 0$ means the system is critical and will reach a non-zero steady state when a neutron source is not present.
  - $\alpha < 0$ implies that the system is subcritical and as $t \to \infty$ the solution will go to zero without a source.
  - There is at most one non-negative eigenvalue.

The time evolution of the angular flux can be used to estimate alpha eigenvalues.

- If we compute $K+1$ time steps of size Δt using a transport solver, we will have the relation

$$\begin{pmatrix} | & | & & | \\ \psi_K & \psi_{K-1} & \cdots & \psi_1 \\ | & | & & | \end{pmatrix} = e^{A\Delta t} \begin{pmatrix} | & | & & | \\ \psi_{K-1} & \psi_{K-2} & \cdots & \psi_0 \\ | & | & & | \end{pmatrix}$$

  or $Y_+ = e^{A\Delta t} Y_-$ and we can take the SVD of Y_- as before.
- Therefore, if we estimate the eigenvalues λ of the $K \times K$ matrix $U^{\mathrm{T}} e^{A\Delta t} U$, we can compute the alpha eigenvalues of the system as

$$\alpha = \frac{\log \lambda}{\Delta t}.$$

- We do not need to a special eigenvalue solver to do this.
- If the time discretization used in the time dependent transport solve is backward Euler (as is common), then a better approximation is

$$\alpha = \frac{\lambda - 1}{\Delta t \lambda},$$

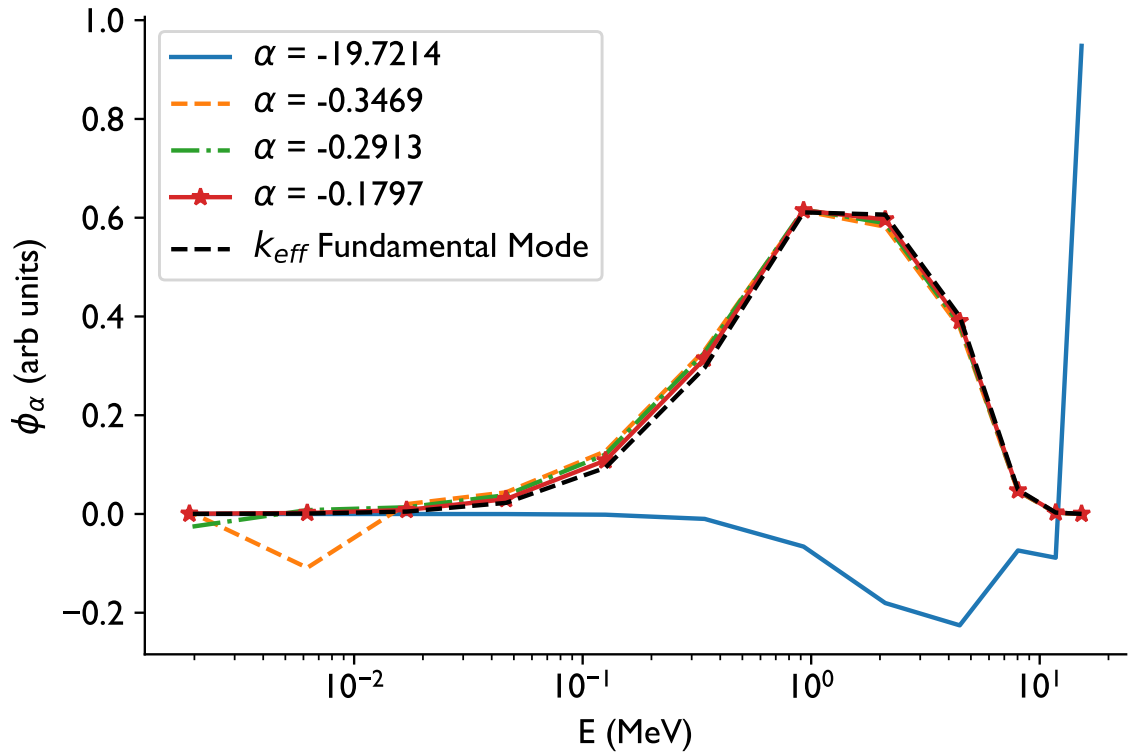
  because this method approximates

$$e^{A\Delta t} \approx (I - A\Delta t)^{-1}.$$

An example problem will allow us to investigate how these alpha eigenvalues behave.

- We consider a 12-group calculation of an infinite medium of Pu-239 with a buckling approximation to simulate a finite sphere.
- We set the radius in the buckling approximation to make $k_{\text{eff}} = 0.95$.
- The eigenvalues we get from the full transport operator for this system are

α (sh^{-1})
-0.179734
-0.291318
-0.346882
-0.48783
-0.756007
-1.33304
-2.64032
-5.52353
-9.00368
-13.7938
-17.4603
-19.7214



Pulsing the sphere with 14.1 MeV neutrons reveals that the DMD approximation computes the eigenmodes present in the system.

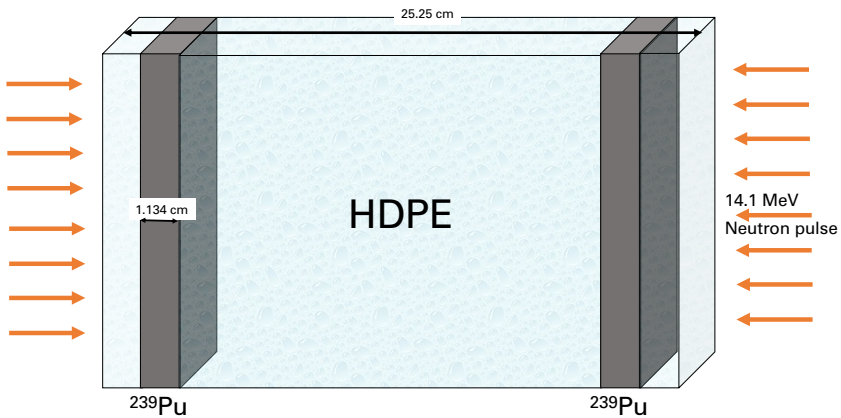
- Using an initial condition of only neutrons in the group containing 14.1 MeV, we run a time dependent problem to a specified final time.
- We then use the last 10 time steps to estimate the α eigenvalues.
- Early in time the modes corresponding to the spike at 14.1 MeV are present, late in time we relax to the fundamental, slowly decaying mode.
- The $\alpha = -0.291318$ sh^{-1} eigenmode is not present because it has a larger number of thermal neutrons than our system does.

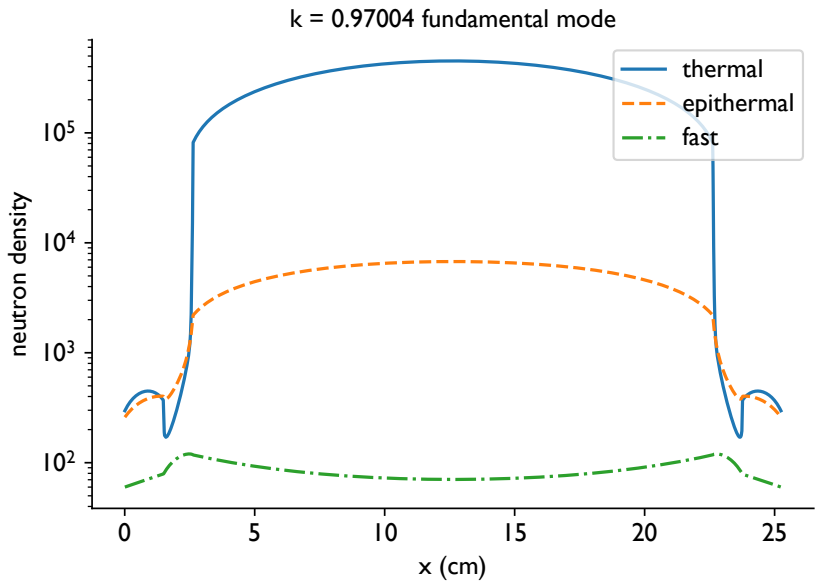
Exact	$t_{\text{final}} = 0.01$ sh	0.2 sh	2 sh	20 sh
-0.179734			-0.182064	-0.179802
-0.291318				
-0.346882			-0.346326	-0.346946
-0.48783				
-0.756007			-0.73261	
-1.33304	-1.40235		-1.5013	
-2.64032			-2.59257	
-5.52353		-5.31228		
-9.00368	-7.65929	-7.60506		
-13.7938				
-17.4603				
-19.7214	-19.7212	-19.7139		

- 1 sh $= 0.01$ μs

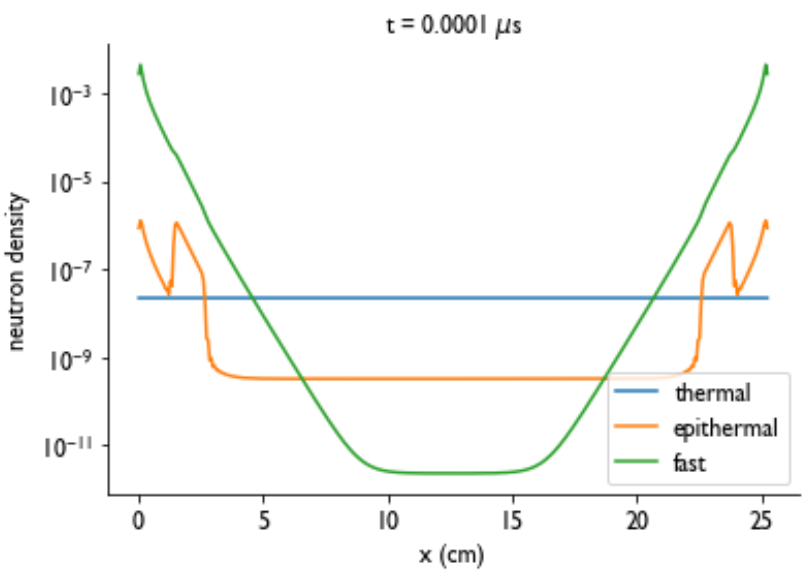
A heterogeneous, subcritical system will test this method on a nontrivial problem.

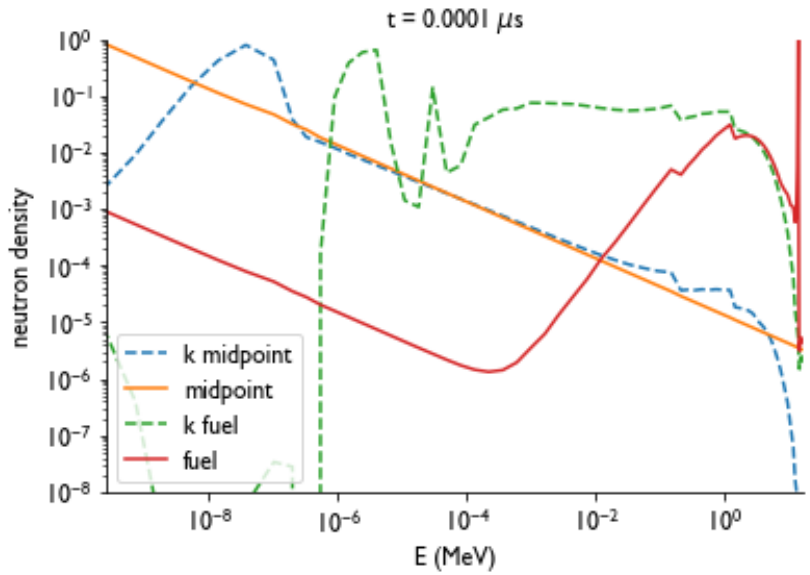
- We consider a slab-geometry problem where a plastic moderator is sandwiched between two slabs of plutonium with a small outer reflector.
- This system will have α eigenvalues associated with the time scale of slow neutrons crossing the moderator.
- The fundamental mode from a *k*-eigenvalue problem has many thermal neutrons in the middle of the problem.
- We consider the situation where DT neutrons enter the slab from both sides at time 0.
- Thermal: $E < 5$ eV, Fast: $E > 0.5 MeV$





In 1 μs the spectrum does not approximate the fundamental k-eigenvalue mode.

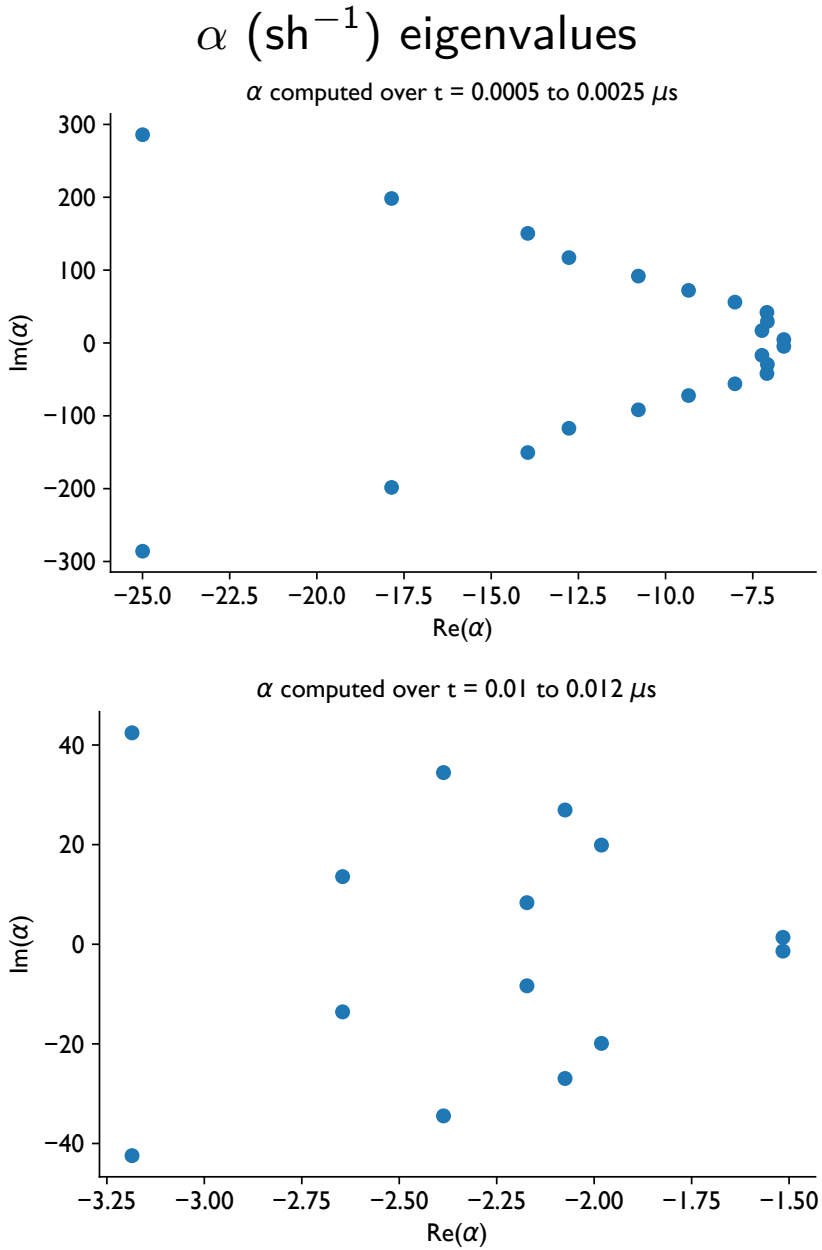


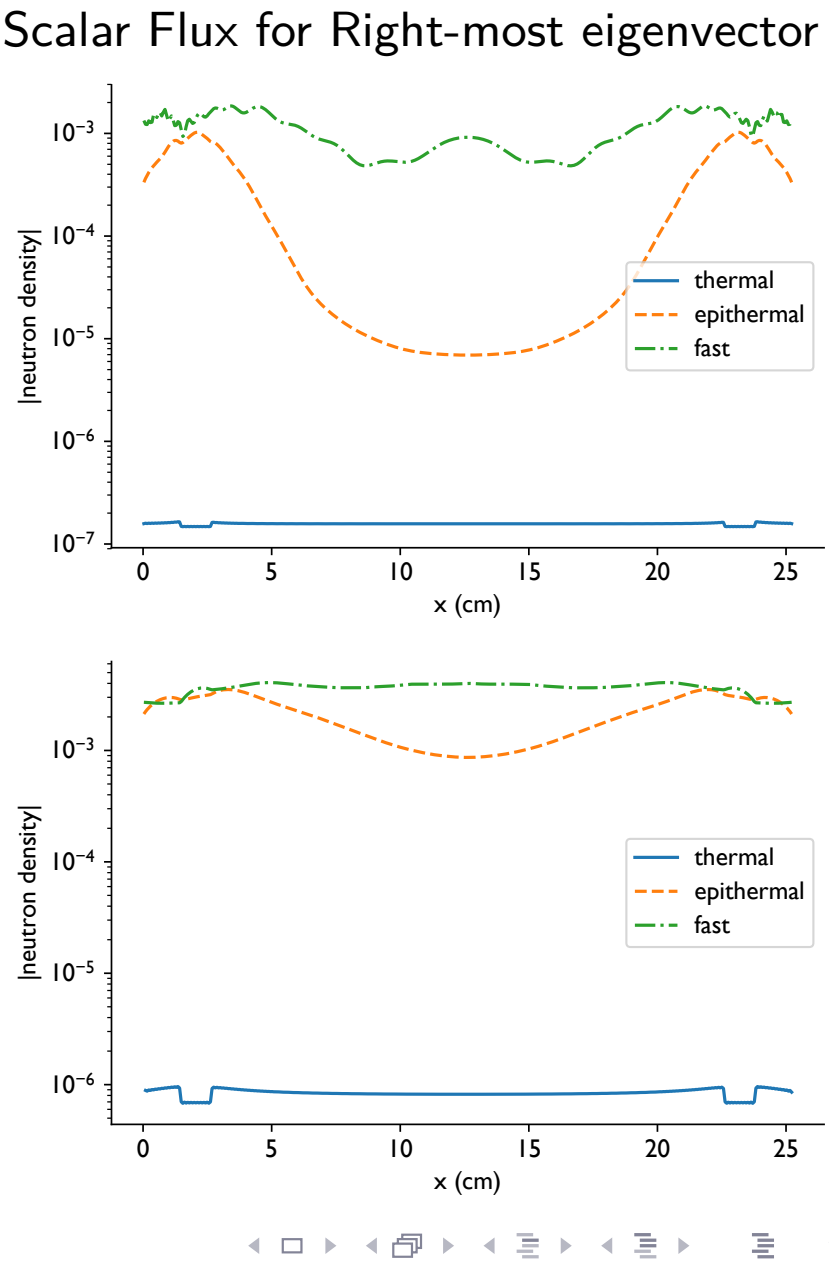


Left: Neutron density (ϕ/v) as a function of space and time.

Right: Neutron spectrum at the center of the HDPE and the center of the fuel.

Using the DMD method at different times finds the predominant eigenmodes present in the system.





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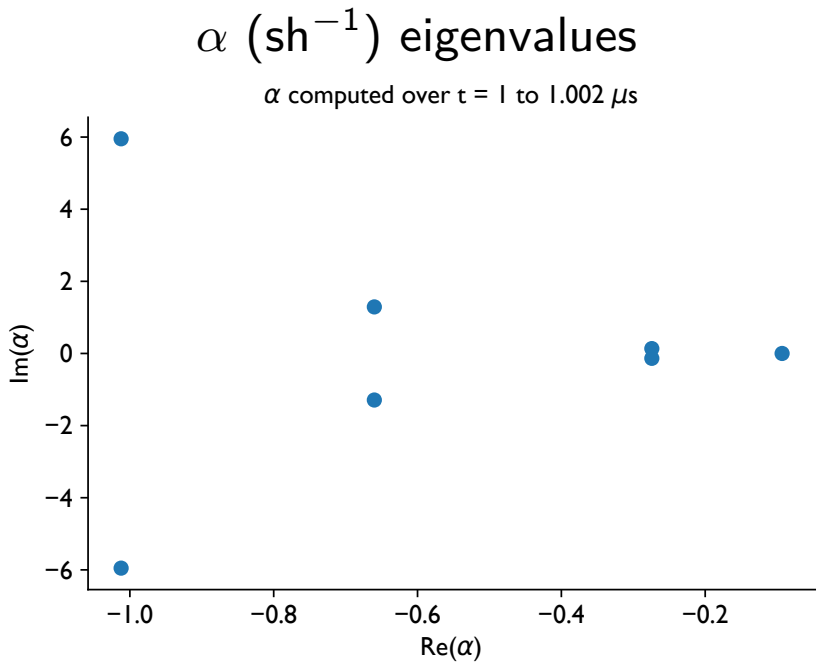
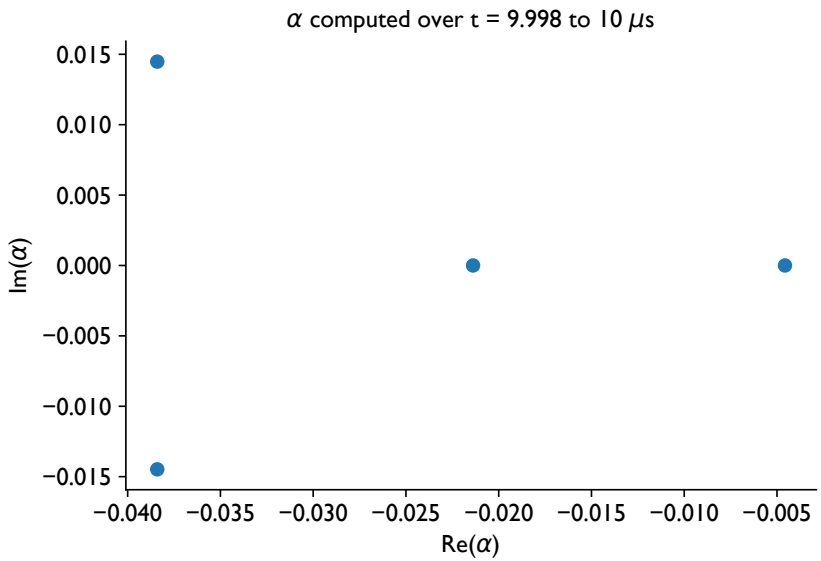
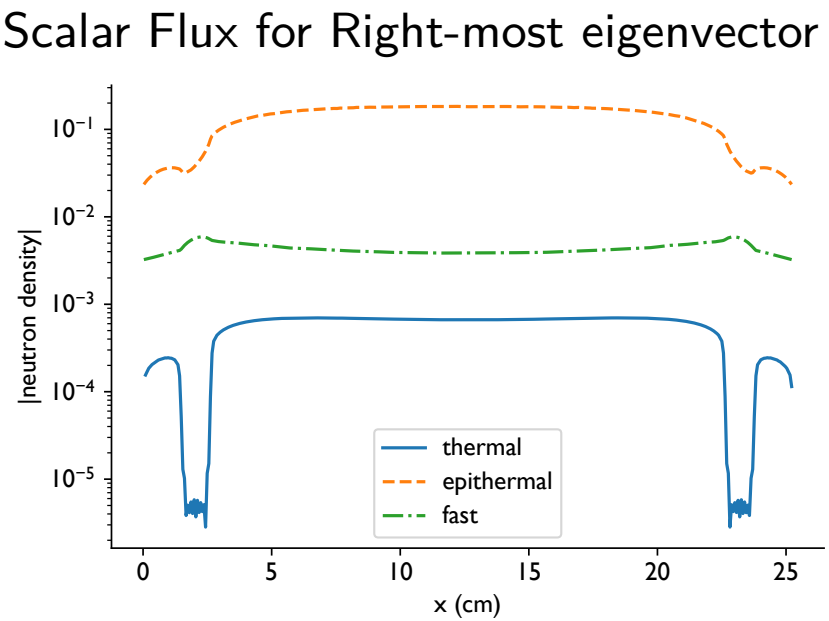
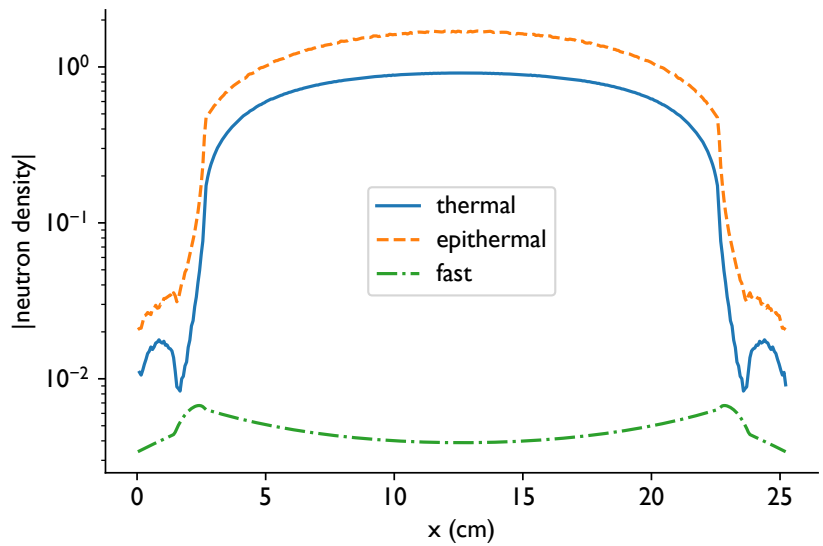








Extensions to the method could be made for more dynamic systems.

- Because the method finds the eigenmodes that are present in the system it will not be susceptible to finding unimportant, slowly decaying modes.
- One can tailor initial conditions to look for certain eigenvalues.
- The DMD approximation can be used for nonlinear operators.
- If the system changed over time we could approximate effective eigenvalues of the changing transport operator:
  - Thermal expansion
  - Depletion/Breeding
- The theoretical interpretation of these eigenvalues is not obvious.

Outline

The approximate operator can be used to find slowly converging modes in an iterative method.

- The discrete ordinates method for transport is typically solved using source iteration (Richardson iteration) and diffusion-based preconditioning/acceleration.
- Source iterations converge quickly for problems with a small amount of particle scattering.
- For strongly scattering media, the transport operator has a near nullspace that can be handled using a diffusion preconditioner.
- However, the question of efficiently preconditioning/accelerating transport calculation on high-order meshes with discontinuous fine elements is an open area of research.
- The approximate operator found from DMD can be used to remove this same near nullspace and improve iterative convergence *without the need for a separate preconditioner or diffusion discretization/solve*.

We consider source iteration for a simple transport problem.

- The steady, single group transport equation with isotropic scattering can be written as

$$L\psi = \frac{c}{4\pi}\phi + \frac{Q}{4\pi},$$

where c is the scattering ratio, Q is a prescribed source, and the streaming and removal operator is

$$L = (\Omega \cdot \nabla + 1).$$

- $\psi(\mathbf{x}, \Omega)$, $\Omega \in \mathbb{S}_2$,

$$\phi(\mathbf{x}) = \int_{4\pi} \psi \, d\Omega = \langle \psi \rangle.$$

- Source iteration solves this problem using the iteration strategy

$$\phi^{\ell} = \left\langle L^{-1} \left(\frac{c}{4\pi}\phi^{\ell-1} + \frac{Q}{4\pi} \right) \right\rangle,$$

where ℓ is an iteration index.

- One iteration is often called a "transport sweep".
- A benefit of source iteration is that the angular flux, ψ does not have to be stored.
- As $c \to 1$, the convergence of source iteration can be arbitrarily slow.

We can write source iteration as a sequence of vectors related by a linear operator.

- Rearranging the transport equation we see that source iteration is an iterative procedure for solving

$$\phi - \left\langle L^{-1} \frac{c}{4\pi} \phi \right\rangle = L^{-1} Q,$$

or

$$(I - A)\phi = b.$$

- Therefore, the source iteration vectors are

$$\phi^{\ell+1} = A\phi^{\ell} + b,$$

or

$$\phi^{\ell+1} - \phi^{\ell} = A(\phi^{\ell} - \phi^{\ell-1})$$

- Therefore, we can cast the difference between iterates in a form that is amenable to the approximation of A using DMD, $Y_+ = AY_-$,

$$Y_+ = \left[\phi^2 - \phi^1, \phi^3 - \phi^2, \ldots, \phi^K - \phi^{K-1}\right],$$

$$Y_- = \left[\phi^1 - \phi^0, \phi^2 - \phi^3, \ldots, \phi^{K-1} - \phi^{K-2}\right].$$

Source iteration can be accelerated by taking several iterates and approximating the solution as $\ell \to \infty$

- As before we define an approximate A as the $K \times K$ matrix:

$$\tilde{A} = U^{\mathrm{T}} A U = U^{\mathrm{T}} Y_{+} V \Sigma^{-1},$$

- We can use $\tilde{A}$ to construct the operator $(I - \tilde{A})^{-1}$ and use this to approximate the solution:

$$\begin{aligned}(I - A)(\phi - \phi^{K-1}) &= b - (I - A)\phi^{K-1} \\ &= b - \phi^{K-1} + (\phi^{K} - b) \\ &= \phi^{K} - \phi^{K-1}.\end{aligned}$$

- The difference $\phi - \phi^{K-1}$ is the difference between step $K-1$ and the converged answer. We define a new vector Δy as the length K vector that satisfies

$$\phi - \phi^{K-1} = U \Delta y. \tag{1}$$

- We then substitute and multiply by U^{T} to get

$$(I - \tilde{A})\Delta y = U^{\mathrm{T}}(\phi^{K} - \phi^{K-1}). \tag{2}$$

  This is a linear system of size K that we can solve to get Δy and then compute the update to ϕ^{K-1} as

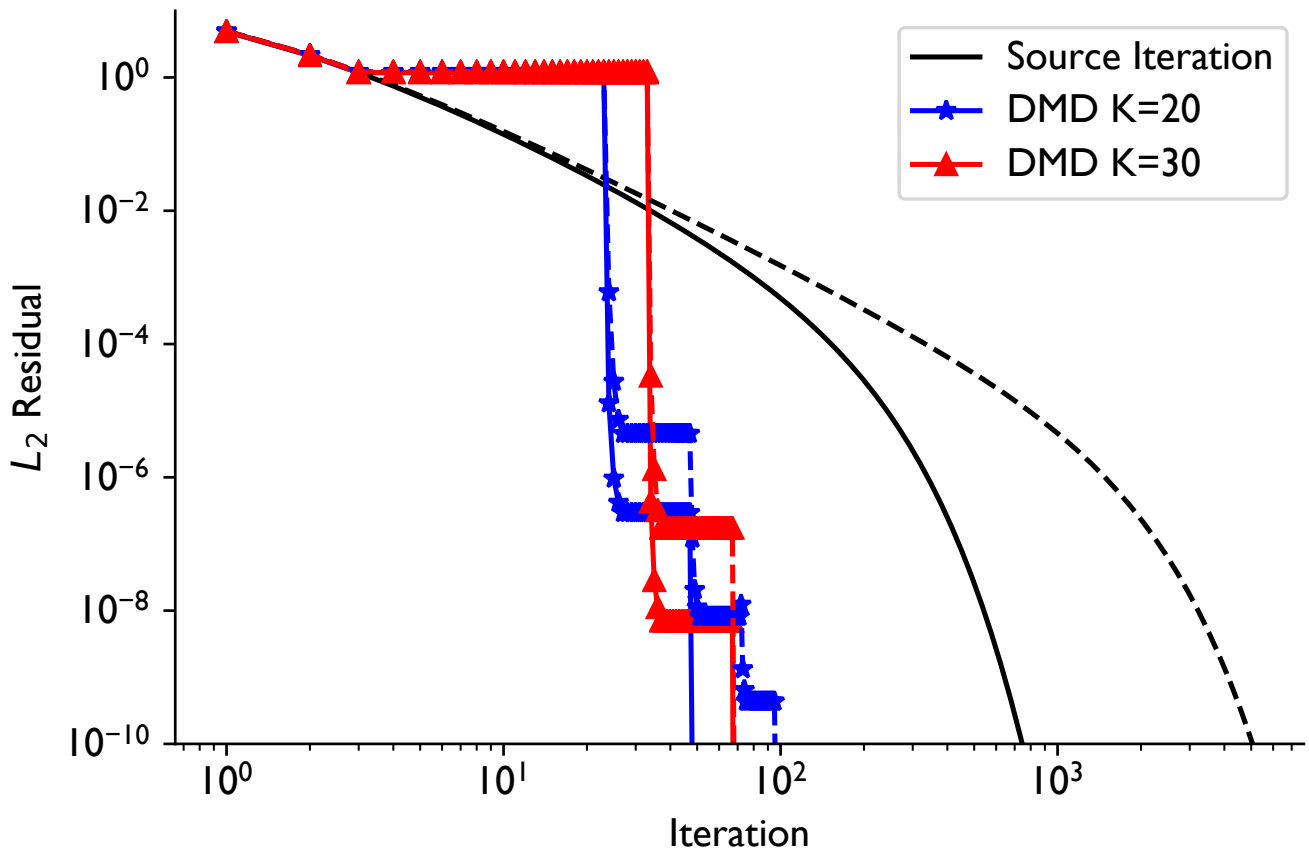
$$\phi \approx \phi^{K-1} + U \Delta y. \tag{3}$$

DMD acceleration requires only source iteration and SVD: no diffusion solver.

- The algorithm is as follows
  1. Perform R source iterations: $\phi^{\ell} = A\phi^{\ell-1} + b$.
  2. Compute K source iterations to form Y_+ and Y_-. The last column of Y_- we call ϕ^{K-1}.
  3. Compute $\phi = \phi^{K-1} + U\Delta y$ as above.
- Each pass of the algorithm requires $R + K$ source iterations.
- The R source iterations are used to correct any errors caused by the approximation of A using the SVD.
- It is easiest to assess convergence between the source iterations.
- This works regardless of the spatial discretization used.
- Other algorithms are possible:
  - Rather than extrapolate to an infinite number of iterations, we can use $\tilde{A}$ to approximate a finite number of source iterations.
  - We could use a coarsened vector $\bar{\phi}$ in the DMD procedure to reduce the memory/computational cost.

DMD works perfectly on a homogenous slab, the ur-demonstration problem for acceleration schemes.

- We consider a slab with vacuum boundaries and a scattering ratio of $c = 0.99$ and 1.0 and 400 spatial zones, S_8 angular discretization, and the diamond difference spatial discretization.
- Solid lines are $c = 0.99$ results and dashed lines are $c = 1.0$



A comparison of the number of iterations as a function of K and c indicates that the convergence is nearly independent of c.

- On the same problem set up, the number of iterations to converge is shown below.

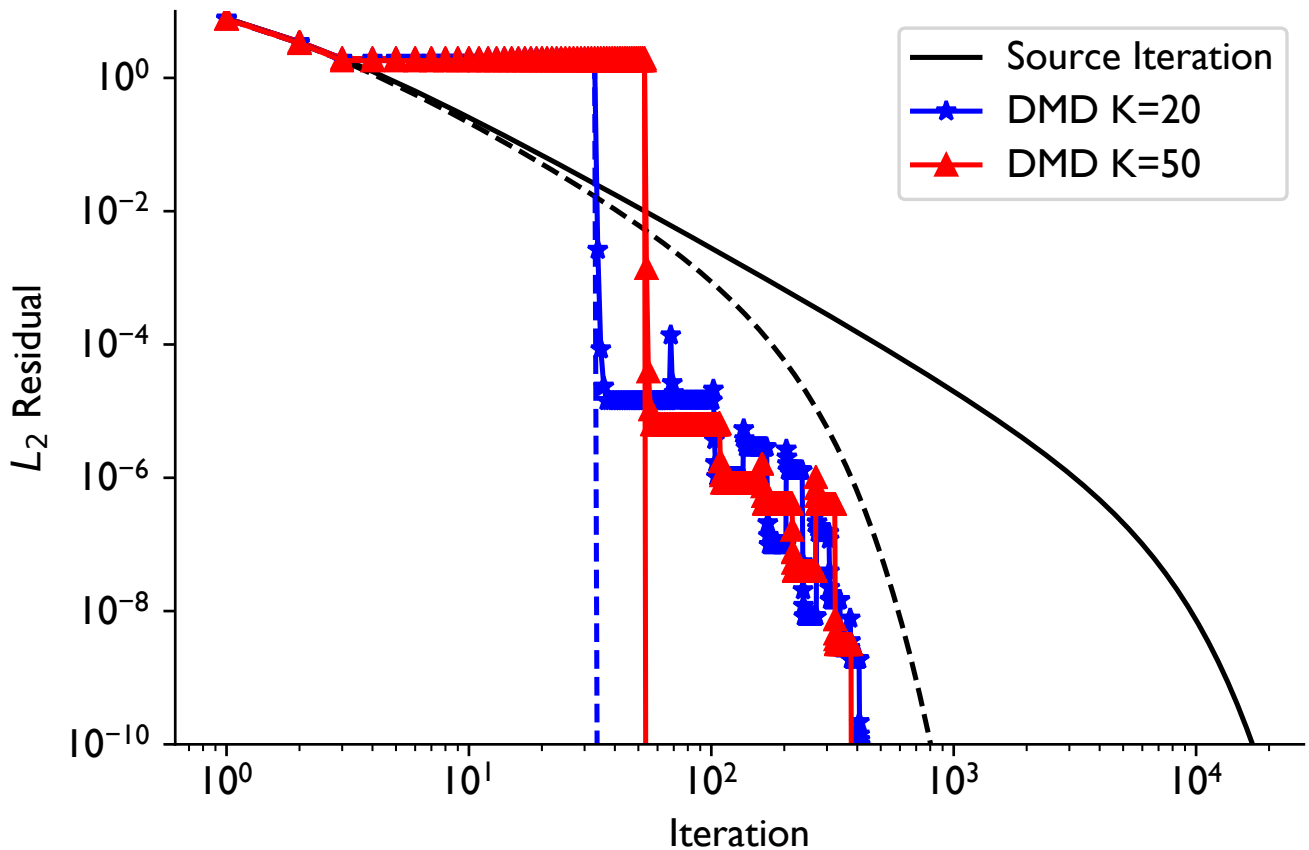
K/c	0.1	0.5	0.9	0.99	0.999	0.9999	0.99999	0.999999
3	8	15	39	70	70	70	70	70
5	10	11	28	90	90	90	90	90
10	15	15	29	60	140	140	140	140
20	25	25	25	49	74	76	76	76
50	55	55	55	56	57	57	57	57
SI	6	17	89	637	2439	3681	3889	3911

Performance does degrade on an insanely heterogeneous problem.

- We consider a problem with vacuum boundaries, 1000 cells, unit domain length, with $c = 0.9999$ and

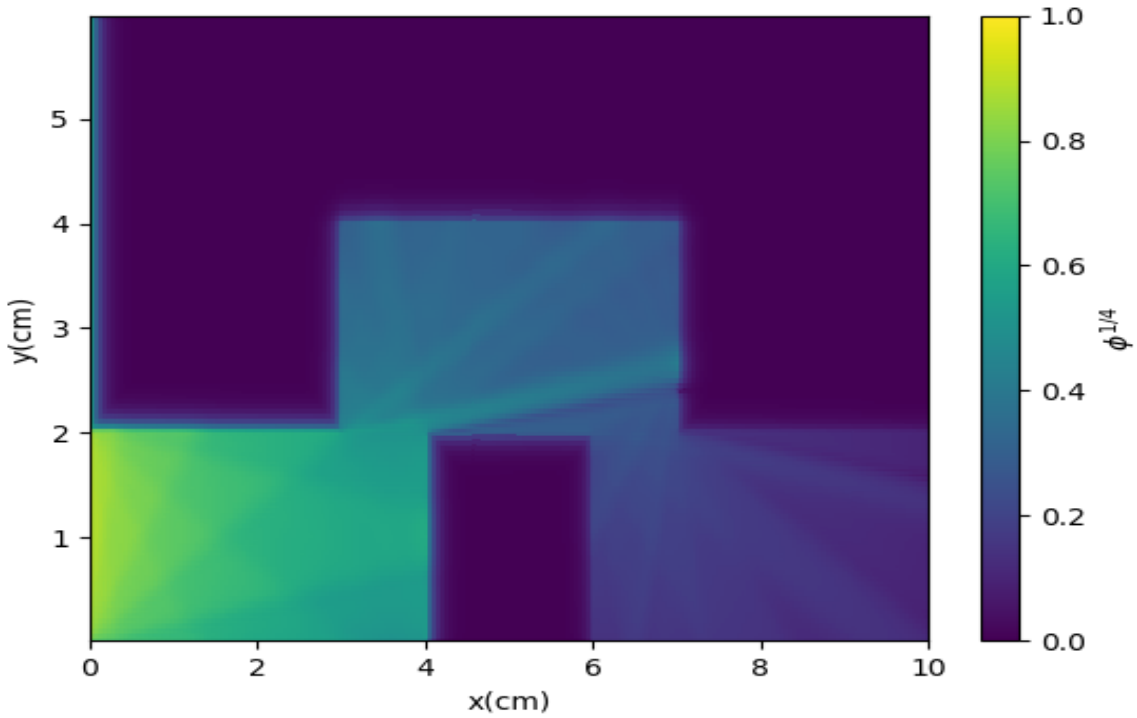
$$\sigma_{\mathrm{t}} = \begin{cases} 2^{p} & \text{cell number odd} \\ 2^{-p} & \text{cell number even} \end{cases}.$$

- Below we see convergence for $p = 5$ (dashed) and $p = 8$ (solid), a factor of about 1000 and 6.5×10^4 between thick and thin cells, respectively.



A version of the crooked pipe problem is a more realistic test.

- We solve a linear, xy-geometry version of the crooked pipe problem where all materials have a scattering ratio of 0.988 (to simulate a realistic sized time step).
- The density ratio between the tick and thin material is 1000.
- Problem solved using fully lumped, bilinear discontinous Galerkin in space and S_8 product quadrature.



The number of iterations required increases slowly with mesh refinement.

- The number of iterations for source iteration and DMD-accelerated calculations with $K = 10$ and $R = 3$.

$(N_x \times N_y)$	DMD	SI
25×15	53	811
50×25	52	873
100×60	78	974
150×90	91	∞_{RML}
200×120	104	∞_{RML}

  ∞_{RML} = functionally infinite on my laptop.

- The increase seems to be the resolution to the 1/2 power (square root of the number of cells per dimension).

There are opportunities to this approach beyond the acceleration strategy outlined above.

- We could use DMD acceleration to compute a low-order transport acceleration (the so-called TSA method). In this case the we would use low-order in angle transport sweeps to estimate the slowly converging modes.
- Additionally, it is possible to estimate $\tilde{A}$ using independently generated vectors. This would enable the $Y_{\pm}$ matrices to be generated using sweeps computed in parallel.
- The big win could be from applying this to other iterative components:
  - Energy group iterations
  - Temperature iterations in radiative transfer.
- The performance of DMD on meshes with cycles is also a possible impact area.

Outline

Is there a way to find an efficient, optimal approximation to time-dependent transport problems?

- This work is a collaboration with Cory Hauck (ORNL) and Martin Frank's group at Karlsruhe Institute of Technology.
- Consider the equation for the evolution of the $m \times n$ matrix $A(t)$:

$$\dot{A} \equiv \frac{d}{dt} A = f(A).$$

- We seek to find a rank r approximation to $A(t)$,

$$Y(t) = U(t) S(t) V(t)^{\mathrm{T}},$$

  where $U(t)$ is $m \times r$, $V(t)$ is $n \times r$ with orthonormal columns and $S(t)$ is $r \times r$ and invertible.
- This looks like an SVD but it is more general because $S(t)$ is not diagonal
- We want to find a dynamic system $\dot{Y}$ such that
  - $\dot{Y}$ lives in the tangent space of rank r matrices,
  - $\|\dot{Y} - \dot{A}\|$ is minimized.

Is there a way to find an efficient, optimal approximation to time-dependent transport problems?

- The evolution equation in the tangent space can be found by differentiating $Y(t)$:

$$\dot{Y} = \dot{U}(t)S(t)V(t)^{\mathrm{T}} + U(t)\dot{S}(t)V(t)^{\mathrm{T}} + U(t)S(t)\dot{V}(t)^{\mathrm{T}}.$$

- We impose the gauge conditions $U^{\mathrm{T}}\dot{U} = 0$, and $V^{\mathrm{T}}\dot{V} = 0$.
- We can operate on $\dot{Y}$ by U^{T} and V to get the following three equations

$$U(t)^{\mathrm{T}}\dot{Y} = \dot{S}(t)V(t)^{\mathrm{T}} + S(t)\dot{V}(t)^{\mathrm{T}},$$

$$U(t)^{\mathrm{T}}\dot{Y}V(t) = \dot{S}(t),$$

$$\dot{Y}V(t) = \dot{U}(t)S(t) + U(t)\dot{S}(t).$$

- These can be rearranged to get

$$\dot{U} = (I - UU^{\mathrm{T}})\dot{Y}VS^{-1}$$

$$\dot{V} = (I - VV^{\mathrm{T}})\dot{Y}^{\mathrm{T}}U(S^{-1})^{\mathrm{T}}$$

$$\dot{S} = U(t)^{\mathrm{T}}\dot{Y}V(t).$$

- The best approximation to $\dot{A}$ can be shown to be

$$\dot{Y} = (I - UU^{\mathrm{T}})\dot{A}VS^{-1} + U(t)^{\mathrm{T}}\dot{A}V(t) + (I - VV^{\mathrm{T}})\dot{A}^{\mathrm{T}}U(S^{-1})^{\mathrm{T}}.$$

There is a splitting of this equation keeps each step in the low rank space.

- Solve the $m \times r$ system over a time step from initial data $Y_0 = U_0 S_0 V_0^{\mathrm{T}}$

$$K_1 = U_0 S_0 + (A(t+\Delta t) - A(t)) V_0,$$

- Compute $U_1 \tilde{S} = K_1$ using QR factorization.
- Solve the $r \times r$ system

$$S^* = \tilde{S} - U_1^{\mathrm{T}} (A(t+\Delta t) - A(t)) V_0,$$

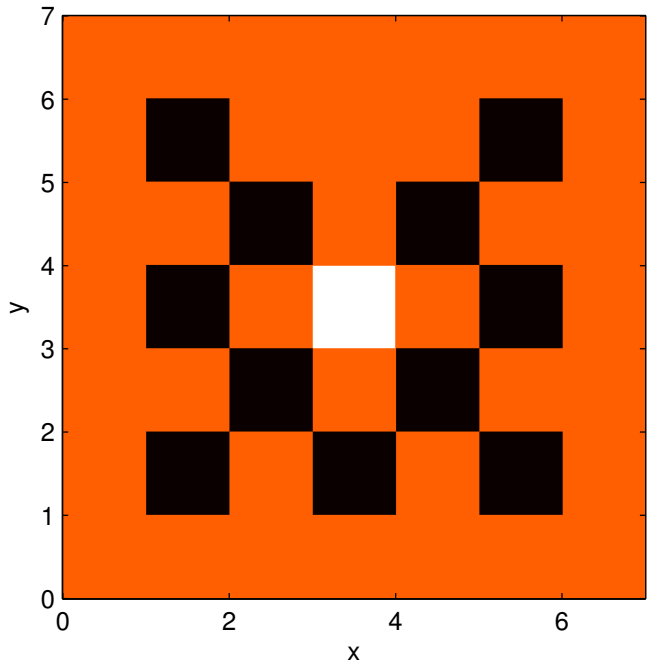
- Solve

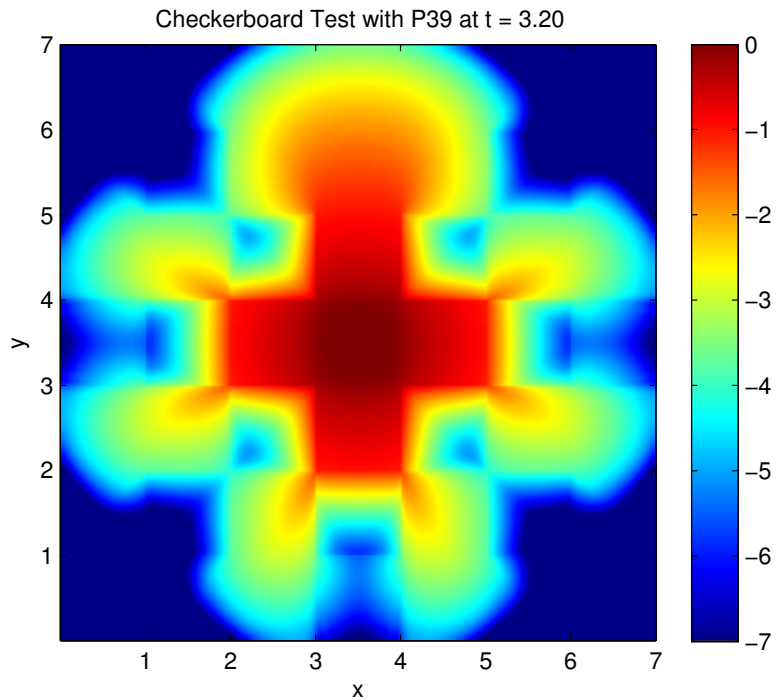
$$L_1 = V_0 (S^*)^{\mathrm{T}} + (A(t+\Delta t) - A(t))^{\mathrm{T}} U_1.$$

- Compute $V_1 S_1^{\mathrm{T}} = L_1$ using QR factorization.
- The value at the end of the step is $Y_1 = U_1 S_1 V_1^{\mathrm{T}}$.
- Each step can be shown to keep the solution in the rank r space.
- This is a first-order algorithm, but higher order algorithms exist.
- To be a truly low rank update, we must be able to estimate $(A(t+\Delta t) - A(t)) = \Delta t \int f(A)\, dt$ without evaluating the full operator.

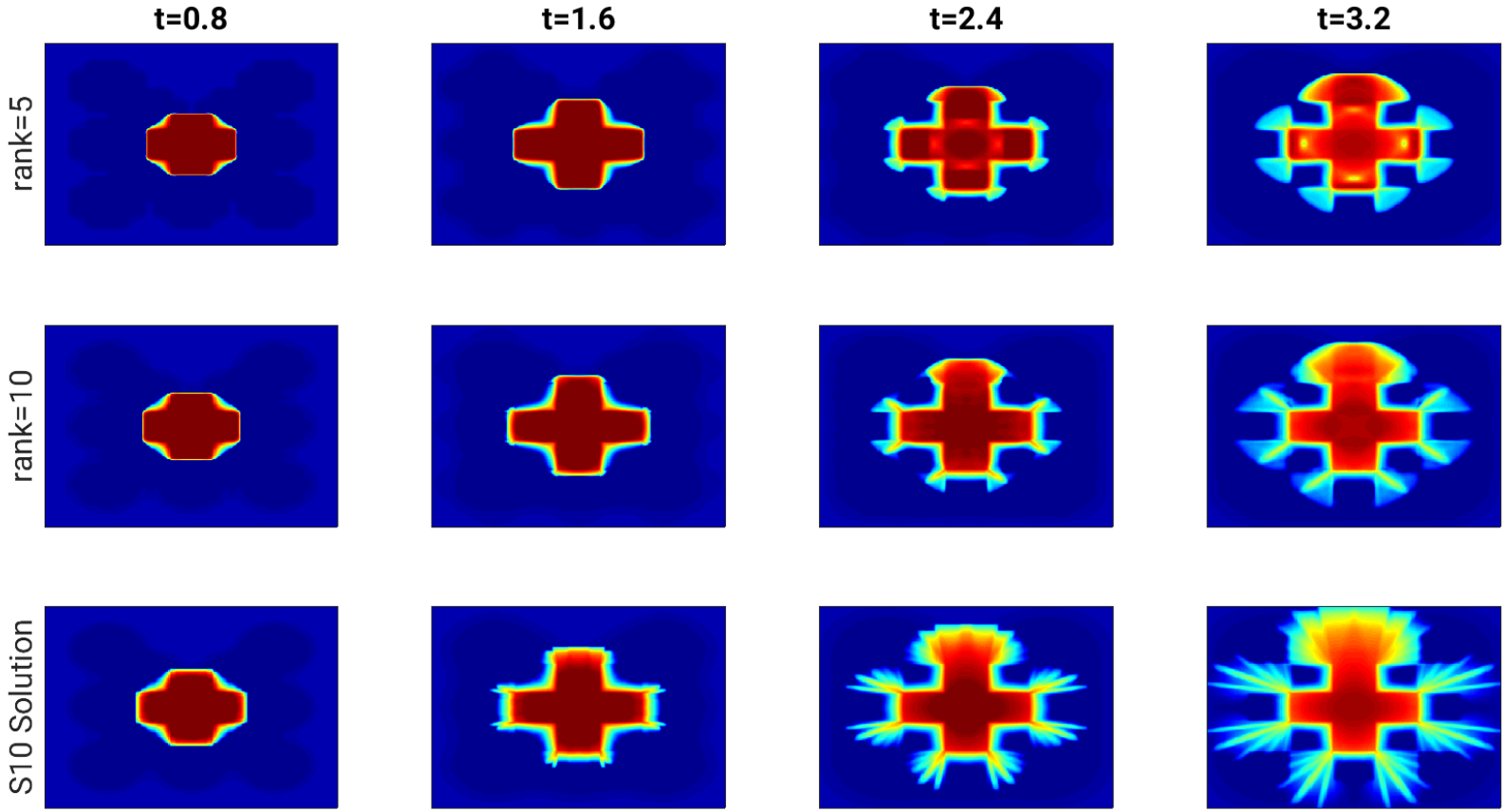
A proof of principle calculation indicates there is merit to this algorithm.

- We consider the S_{10} solution to Brunner's lattice problem and use the full transport operator to estimate $(A(t+\Delta t) - A(t))$.
- The data has 280×280 degrees of freedom in space and 100 degrees of freedom in angle.
- Figure below has pure scatter in orange, pure absorber in black, and a source in white. A solution with a logarithmic color scale is on the right.

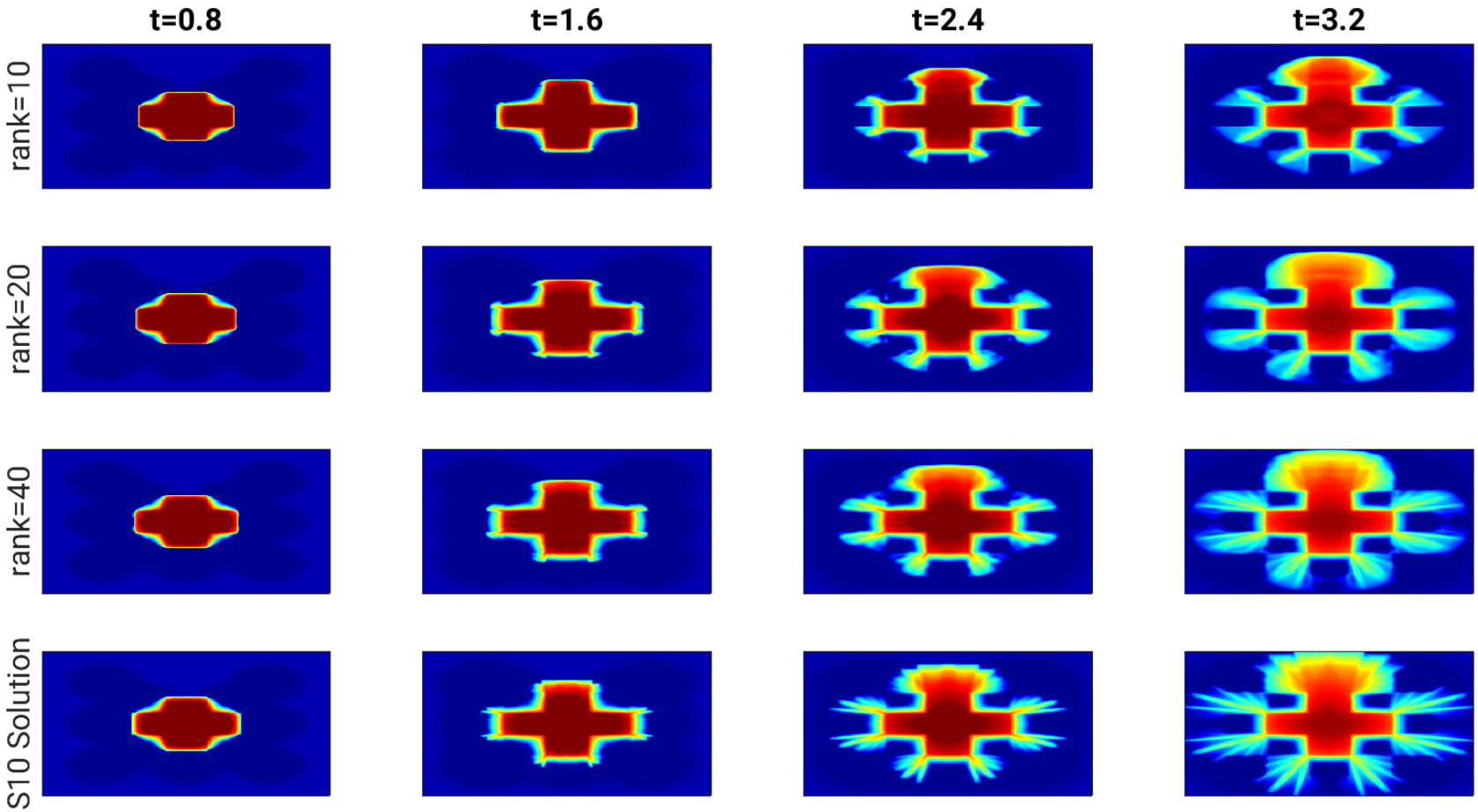




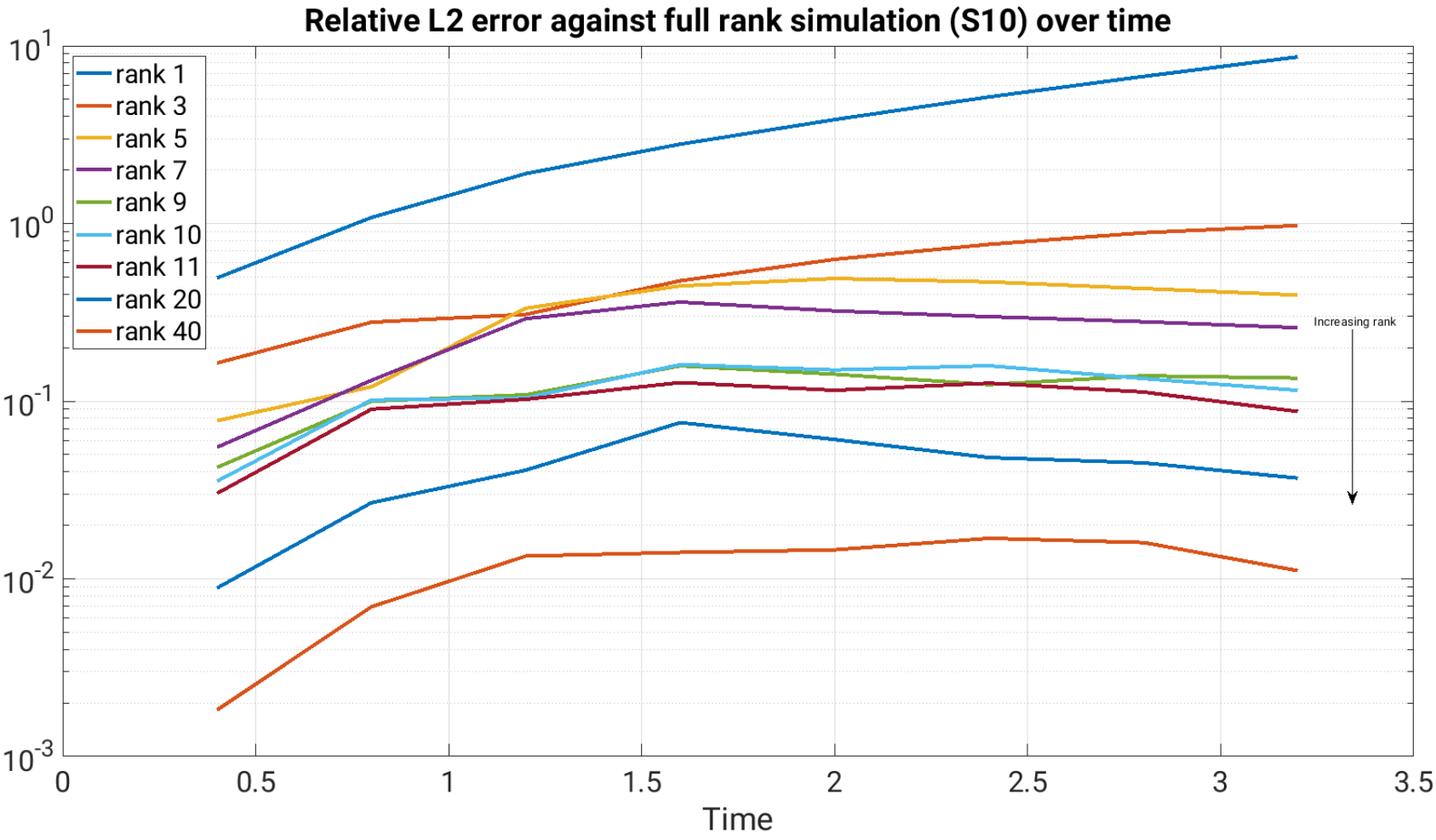
The low-rank comparisons give promising results on this problem.



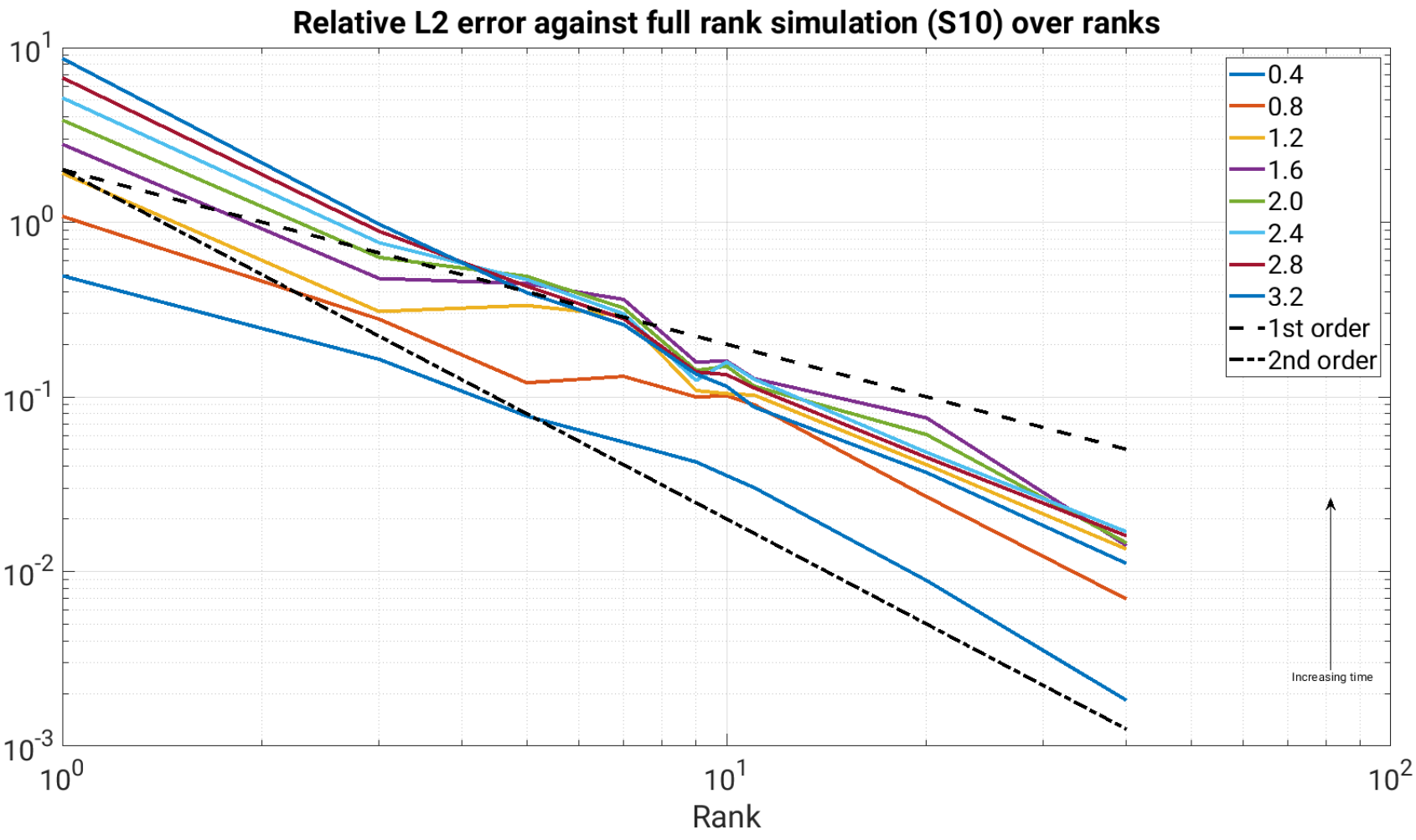
The low-rank comparisons give promising results on this problem.



The low-rank approximations are better at early time.



The error decreases at second-order in the rank beyond rank 10.



We need to derive the low-rank approximations to the update.

- As mentioned before, if we use the full system to approximate $(A(t+\Delta t) - A(t))$ there is no savings.
- We need to express the projections $U_1^{\mathrm{T}}(A(t+\Delta t) - A(t))V_0$, $U_1^{\mathrm{T}}(A(t+\Delta t) - A(t))V_0$, and $(A(t+\Delta t) - A(t))^{\mathrm{T}}U_1$ in terms of the discretizations in space and angle.
- Then the update will only require the smaller memory footprint of the low rank operators during the update.
- We have derived the equations for 1-D slab geometry, but do not have solutions yet.

Outline

Data-Driven algorithms are widely applicable in transport problems.

- Using a DMD approach to compute approximate operators gives one the ability to
  - Estimate eigenvalues for the system, and
  - Accelerate calculations.
- Furthermore, we can use similar techniques to derive a low-rank approximation to the dynamics of transport.
- There is much further research to be done, but progress is exciting.

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