ONE-DIMENSIONAL MODELS FOR TRANSPORT IN SOLID CYLINDERS

Ryan G. McClarren
University of Notre Dame

Alex R. Long
Los Alamos National Laboratory
Admittedly, It Is An Incomplete Bridge

The so-called Marshak wave problem is a classic problem in high-energy density radiative transfer.

- Several variations of the problem exist, but the main theme is that a cold medium is subject to a source of radiation and a wave propagates through the medium.
- The problem usually imagines a slab geometry configuration.

One-dimensional models for particle transport in evacuated ducts has been used to model the transport of particles using a simplified model.

- The transport in the duct is modeled using an effective scattering from the duct walls.

There is active research in developing models for Marshak waves in configurations reminiscent of the duct problem.
THE MARSHAK WAVE PROBLEM

The Classical Work

- The Marshak wave problem has a cold slab with an heat source applied at a particular point.
- Via radiative transfer the medium is heated.
- If the material is optically thick, as the original works assume, the governing equations is a nonlinear diffusion equation.
  \[ \frac{\partial e(T)}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{3\kappa(T)} \frac{\partial}{\partial x} acT^4 \right) \]
- Petschek demonstrated how to find similarity solutions.
- Larsen and Pomraning showed that this is the asymptotic limit of a non-equilibrium system.
SIMILARITY SOLUTION

Power Law Opacities

- If the opacity has the form
  \[ \kappa(T) = \kappa_0 T^{-n} \]
- There exists a similarity solution \( T(\xi) \) where
  \[ \xi = C(n) \frac{x}{\sqrt{t}} \]
- The steepness of the wavefront is controlled by the power in the opacity.

The similarity solution for a power-law opacity, from Nelson and Reynolds, LA-UR-09-04551.
**Hammer and Rosen extended the theory to include a non-constant driving temperature.**

**Hurricane and Hammer tried to account for radial losses in the model.**

**Use observation that away from wave front the material temperature changes slowly.**

**Other work added radiation energy density terms and material motion into the model.**
The Hurricane-Hammer model points out that behind the wave-front the time
dependence of the solution is negligible.

This makes the problem here an 2-D Laplace equation.

\[ \nabla^2 T^4 = 0 \]

In 2-D Cartesian geometry, with \( x \) being the direction that the wave is
propagating, the wavefront will then be a linear combination of cosines, that
determine the curvature of the wavefront.

Detailed analysis then gives a form for the curvature as a function of boundary
conditions, etc., and an equation for the propagation of the wavefront.

Hammer and Rosen also use the slow change behind the wavefront, but they
use this to derive the results from changing boundary temperatures for 1-D
problems, and solve problems regarding supersonic and subsonic diffusion
waves.
COMPARISONS WITH EXPERIMENT SHOW THESE MODELS ARE REASONABLE

Improvement Is Possible

- A recent paper by Moore, et al. compares Marshak waves created at NIF with the different models of Hammer and Hurricane.
- There seems to be a gap between analytic theory and full-blown radiation hydrodynamics simulations.
- Is a simple, ordered, approximation to the radiative transfer in the experiment possible?
MEANWHILE, 1-D MODELS WERE BEING DEVELOPED FOR TRANSPORT IN A DUCT

Sometimes By The Same People

- Consider particles traveling in a duct of material that is surrounded by a wall that can reflect a fraction, $c$, of the particles back into the duct.
- Most of the models were developed for evacuated ducts where particles enter at one end.
- In an evacuated duct, the transport equation is written as
  \[ \mu \frac{d\Psi}{dz} + \sqrt{1 - \mu^2} \left( \cos \gamma \frac{d\Psi}{dx} + \sin \gamma \frac{d\Psi}{dy} \right) = 0 \]
- Using a Galerkin procedure, a simplified 1-D model can be derived
  \[ \mu \frac{d\Psi}{dz} + \sqrt{1 - \mu^2} A \Psi = \frac{2c}{\pi} B \int_{-1}^{1} \sqrt{1 - \mu'^2} \Psi(z, \mu') d\mu' \]
  \[ \psi(x, y, z, \mu, \gamma) = \sum_{n=1}^{N} b_n(x, y, \gamma) \Psi_n \]
EXTENSIONS TO THIS DUCT WORK

- A quadratic expansion was introduced by Garcia, et al. (3 basis functions).
- Multigroup models for the transport also exist.
- Prinja modified the model to allow particles to re-enter the duct at different places (non-local reflection).
- Many papers on the efficient solution of the models.
- Models used in shielding and acoustics.
WALL MIGRATION OF NEUTRONS

As a result, particles can re-enter duct ahead of or behind where the particle left the duct.

Neutrons collisions don’t take place at the wall, rather inside the wall.

Wall-Scattering now has a non-local kernel

\[ B \int_{0}^{Z} dz' \int_{-1}^{1} (1 - \mu'^2)^{\frac{1}{2}} K(z' \rightarrow z) \tilde{\psi}(z, \mu') d\mu'. \]

Example kernel function

\[ \int_{0}^{Z} K(z' \rightarrow z) dz' = \int_{0}^{Z} \frac{\lambda}{2} \exp(-\lambda|z - z'|) dz' \]
THE CYLINDER IN THE MARSHAK WAVE PROBLEM IS A DUCT

It Is Just Not An Evacuated Duct

- The radiative transfer in a cylinder is, in some sense, the opposite of the duct model.
  - Collisions take place in the cylinder and the walls have a reflection probably that could be zero.

- The 1-D duct models make no assumptions that the walls must scatter particles back into the domain.

- Furthermore, as pointed out early on in their history, there is no limitation to extending to non-evacuated ducts in the models.
  - They are just typically applied to evacuated regions.

- The curvature in the wave-front, time-dependence of drive, etc. could be included in the model.

- This would be a simplified model that captures the effects the 1-D diffusion models lack.
WE DEVELOP A MODEL FOR TIME-DEPENDENT TRANSPORT IN A SOLID CYLINDER

- We consider a linear transport problem:

\[
\frac{1}{v} \frac{\partial \Psi}{\partial t} + (1 - \mu^2)^{\frac{1}{2}} \left( \cos \varphi \frac{\partial \Psi}{\partial x} + \sin \varphi \frac{\partial \Psi}{\partial y} \right) + \mu \frac{\partial \Psi}{\partial z} + \sigma_t \Psi = \frac{\sigma_s}{4\pi} \int_{-1}^{1} d\mu' \int_{0}^{2\pi} d\varphi' \Psi(x, y, z, \mu', \varphi', t) + \frac{Q}{4\pi}
\]

\[
\Psi(x, y, 0, \mu, \varphi, t) = g_l(x, y, \mu, \varphi, t), \quad h(x, y) < 0, \quad \mu > 0,
\]

\[
\Psi(x, y, Z, \mu, \varphi, t) = g_r(x, y, \mu, \varphi, t), \quad h(x, y) < 0, \quad \mu < 0.
\]

\[
\Psi(x, y, z, \mu, \varphi, t) = 0, \quad h(x, y) = 0, \quad \omega \cdot \mathbf{n} < 0,
\]

\[
\omega = (\cos \varphi, \sin \varphi, 0)
\]

- We then propose an expansion in basis functions that contain the variation in \(x, y,\) and the azimuthal angle.

\[
\Psi(x, y, z, \mu, \varphi) \approx \sum_{i=1}^{I} \psi_i(z, \mu, t)b_i(x, y, \varphi).
\]

Here \(I \leq 3.\) The basis functions are

\[
b_1(x, y, \varphi) = 1,
\]

\[
b_2(x, y, \varphi) = u[D(x, y, \varphi) - v],
\]

\[
b_3(x, y, \varphi) = r[D(x, y, \varphi) - v][D(x, y, \varphi) - v - q] - r/u^2.
\]
THE GALERKIN PROJECTION TO A 1-D MODEL

- The function D gives the distance from a point to the edge of the cylinder along the direction \( \omega \). For a cylinder this is

\[
D(x, y, \varphi) = r \cdot \omega + [(r \cdot \omega)^2 + r^2 - x^2 - y^2]^\frac{1}{2}, \quad \mathbf{r} = (x, y, 0)
\]

- The basis expansion is substituted into the transport equation, and the resulting equation is integrated against the basis functions to get

\[
\frac{1}{v} \frac{\partial \psi_i}{\partial t} + \mu \frac{\partial \psi_i}{\partial z} + \sigma_t \psi_i + \sqrt{1 - \mu^2} \sum_{j=1}^{3} a_{ij} \psi_j(z, \mu, t) = \frac{\sigma_s}{2} \phi_i(z, t) + Q_i,
\]

where

\[
Q_i = \frac{1}{2\pi A} \int_R dx \int_0^{2\pi} d\varphi \int_0^{2\pi} d\varphi \; b_i(x, y, \varphi) \frac{Q}{4\pi}.
\]

- These equations are a set of coupled 1-D transport equations. The effective total cross-section depends on \( \mu \).

- The coupling does make the equations look like a multigroup system.
THE MATRIX ELEMENTS ARE NOT NECESSARILY POSITIVE

From Garcia, Ono, And Veira (2000)

\[ a_{11} = b_{11} = \frac{L}{(\pi A)} , \]
\[ a_{12} = b_{12} = u - uvL/(\pi A) , \]
\[ a_{21} = b_{21} = -uvL/(\pi A) , \]
\[ a_{22} = u^2v^2L/(\pi A) , \]
\[ b_{22} = -uv[u - uvL/(\pi A)] . \]
\[ a_{13} = b_{13} = -qr + (v^2 + qv - 1/u^2)rL/(\pi A) , \]
\[ a_{31} = b_{31} = (v^2 + qv - 1/u^2)rL/(\pi A) , \]
\[ a_{23} = (2r/u) - (v^2 + qv - 1/u^2)uvrL/(\pi A) , \]
\[ b_{23} = uv[qr - (v^2 + qv - 1/u^2)rL/(\pi A)] , \]
\[ a_{32} = -(v^2 + qv - 1/u^2)uvrL/(\pi A) , \]
\[ b_{32} = r(v^2 + qv - 1/u^2)[u - uvL/(\pi A)] , \]
\[ a_{33} = (v^2 + qv - 1/u^2)^2r^2L/(\pi A) , \]

and
\[ b_{33} = -r(v^2 + qv - 1/u^2) \]
\[ \times [qr - (v^2 + qv - 1/u^2)rL/(\pi A)] . \]

\[ q = 8\rho \left[ \frac{9\pi}{5} (9\pi^2 - 64)^{-1} - \frac{2}{3\pi} \right] \]
\[ r = \rho^{-2} \left[ 1 - \frac{576}{25} (9\pi^2 - 64)^{-1} \right]^{-1/2} \]
\[ u = \left\{ \frac{1}{2\pi A} \int_0^{2\pi} \int_R [D(x, y, \omega) - v]^2 \, d\varphi \, dx \, dy \right\}^{-1/2} \]
\[ v = \frac{1}{2\pi A} \int_0^{2\pi} \int_R D(x, y, \omega) \, d\varphi \, dx \, dy . \]
WE SOLVE THE EQUATIONS WITH A DISCRETE ORDINATES SCHEME

Off-Diagonal Terms Are Updated Through Source Iteration

• We compute transport sweeps as

\[ \mu_k \frac{\partial}{\partial z} \psi_{ik}^\ell + (\sigma_t^* + a_{ii}) \psi_{ik}^\ell = \sum_{i \neq j} a_{ij} \sqrt{1 - \mu_k^2 \psi_{jk}^{\ell-1}} + \frac{\sigma_s}{2} \phi_{i}^{\ell-1} + Q_{ik}^* \]

• \( k \) is the angle index, and superscript \( \ell \) is the iteration index. The asterisks denote changes made to write the time-dependent equation in quasi-steady form.

\[ \sigma_t^* = \sigma_t + \frac{1}{v \Delta t} \quad Q_{ik}^* = Q_i + \frac{\psi_{ik}^{\text{old}}}{v \Delta t} \]

• One could imagine developing acceleration strategies for these equations, but we have not needed them in our work to date.
STeady-State Results
Comparison With Monte Carlo

- We solve a problem of a cylinder of a single material with unit total cross-section and length 10.
- We vary the radius between 2 and 6, and the scattering ratio from 0 to 1.
- A unit incident, isotropic flux is imposed at z=0.
- Comparisons are made with calculations from Milagro from LANL.
- Contour maps will have MC results at positive r and 1-D results at “negative” r.
1-D MODELS PERFORM BETTER WITH LOW SCATTERING RATIO

3 Basis Functions

Radius 6

Radius 2
THE QUADRATIC MODEL IS NECESSARY
2 And 1 Basis Functions Are Significantly Worse

2-Basis Functions, Radius 6

1-Basis Function, Radius 6
3 BASIS FUNCTION SOLUTIONS VERSUS RADIUS

- **c = 0.25**
- **c = 0.75**
- **c = 0.9**
- **c = 1**
BIGGEST DISCREPANCY IS NEAR RADIAL EDGE

3 Basis Function Solutions, R=6

\( c = 0.25 \)

\( c = 0.75 \)

\( c = 0.9 \)

\( c = 1 \)
TIME-DEPENDENT SOLUTIONS HAVE THE OPPOSITE PROBLEM

3 Basis Functions Solutions At 1 Mean-Free Time Move Too Fast

Radius 6

Radius 2
EXTENDING MODEL TO RADIATIVE TRANSFER PROBLEMS WILL INVOLVE COMPLICATIONS

Non-Constant Cross-Sections

• Beyond adding the coupling to the material temperature equation, there are additional complications to solving the 2-D Marshak wave problem in a cylinder.

• Spatially dependent cross-sections will cause create additional coupling between the angular fluxes.

• In principle, these can all be handled.

• An interesting question is the diffusion limit of this model.

• Scaling the equations so that scattering is large, and absorption, time-dependence, sources, and the correction terms are small, I get that the leading order solution satisfies

\[
\frac{1}{v} \frac{\partial \phi_i^{(0)}}{\partial t} - \frac{\partial}{\partial z} \left( \frac{1}{3\sigma_t} \frac{\partial \phi_i^{(0)}}{\partial z} \right) + \sigma_a \phi_i^{(0)} + \frac{\pi}{4} \sum_{j=1}^{3} a_{ij} \phi_j^{(0)} = 2Q_i.
\]