The Spectral Volume Method as Applied to Transport Problems

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Outline

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  - In regions with constant cross-sections, the transport solution is often smooth.
  - In such regions high-order reconstructions using large cells can be more efficient.
- Extra-unknowns, if they don't increase the communication burden, might only marginally increase the computational cost.

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- This can be thought of as a generalization of the simple corner balance and other sub-cell balance method previously presented.
- The term spectral is used here to note that the solution in each cell is reconstructed via polynomials in a similar way to a spectral method on a finite domain.

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- This is the topic of ongoing work.

Derivation of the method

- We begin with the steady-state transport equation in slab-geometry, using discrete ordinates:

$$\mu_l \partial_x \psi_l + \sigma_{\mathrm{t}} \psi_l = \frac{\sigma_{\mathrm{s}}}{2} \langle \psi_l \rangle + \frac{Q}{2}, \qquad l \in [1, L].$$

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- We then further partition each cell i into K sub-cells with width $\Delta x_{i,k}$.
- Averaging over a generic sub-cell k of cell i yields

$$\frac{\mu_l}{\Delta x_{i,k}} \left(\hat{\psi}_l^{i,k+1/2} - \hat{\psi}_l^{i,k-1/2} \right) + \sigma_\mathrm{t} \psi_l^{i,k} = \frac{\sigma_\mathrm{s}}{2} \langle \psi_l^{i,k} \rangle + \frac{Q^{i,k}}{2},$$

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- This polynomial is given by

$$p_l(x) = \sum_{k=1}^{K} \varphi_k(x)\psi_l^{i,k},$$

where

$$\varphi_k(x) = \prod_{q=1,q\neq k}^{K} \frac{x - x_{iq}}{x_{i,k} - x_{iq}}.$$

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- We then use this polynomial to give the value of the ψ_l inside of cell i. Specifically, this polynomial gives the interfacial values between the sub-cells:

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- At the interface between cells we use the principle of upwinding to choose the value. Specifically,

$$\hat{\psi}_l^{i,k-1/2} = \begin{cases} p_l(x_{i-1/2}) & \mu_l < 0 \\ p_l^{i-1}(x_{i-1/2}) & \mu_l > 0 \end{cases} \quad \text{for } k = 1,$$

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- Using Gauss-Lobatto quadrature points to define the sub-cell edges has been shown to maintain convergence.
- For a generic cell this approach leads to

$$x_{i,k+1/2} = \frac{\Delta x_i}{2}\left(1 - \cos\left(\frac{k\pi}{K}\right)\right), \quad k = 0, \ldots, K.$$

Sub-cell Edges

- The edges when define with the GL quadrature points are

K	Points
2	0, 0.5, 1
3	0, .25, .75, 1
4	0, 0.146447, 0.5, 0.853553, 1
5	0, 0.0954915, 0.345492, 0.654508, 0.904508, 1
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- Notice how these points are clustered near the edges of the main cell.

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- This will be the subject of future work.

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- If we multiply the balance equation for a sub-cell by $\Delta x_{i,k}/\Delta x_i$ and sum over $k = 1 \ldots K$, we get

$$\frac{\mu_l}{\Delta x_i}\left(\hat{\psi}_l^{i,K+1/2} - \hat{\psi}_l^{i,1/2}\right) + \sigma_\mathrm{t} \sum_{k=1}^{K} \frac{\Delta x_{i,k}}{\Delta x_i} \psi_l^{i,k}, = \frac{\sigma_\mathrm{s}}{2} \langle \sum_{k=1}^{K} \frac{\Delta x_{i,k}}{\Delta x_i} \psi_l^{i,k} \rangle + \frac{\bar{Q}^i}{2},$$

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- To show conservation over the entire domain, we multiply this equation by Δx_i and sum over all cells to get

$$-\mu_l\left(\hat{\psi}_l^{I,K+1/2} - f_l\right) + \sum_i \Delta x_i \left(\frac{\sigma_s}{2}\langle \bar{\psi}_l^i \rangle - \sigma_t \bar{\psi}_l^i + \frac{\bar{Q}^i}{2}\right) = 0, \quad \mu > 0,$$

and

$$-\mu_l\left(g_l - \psi_l^{1,1/2}\right) + \sum_i \Delta x_i \left(\frac{\sigma_s}{2}\langle \bar{\psi}_l^i \rangle - \sigma_t \bar{\psi}_l^i + \frac{\bar{Q}^i}{2}\right) = 0, \quad \mu < 0.$$

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- We can see this by writing out the equations for $\mu > 0$ for a generic cell with $K = 2$ and $\Delta x_i = 2\Delta x_{i,k}$:

$$\frac{\mu_l}{\Delta x_i}\left((\psi_l^{i,2} + \psi_l^{i,1}) - (3\psi_l^{i-1,2} - \psi_l^{i-1,1})\right) + \sigma_{\mathrm{t}}\psi_l^{i,1} = \frac{\sigma_{\mathrm{s}}}{2}\langle\psi_l^{i,1}\rangle + \frac{Q^{i,k}}{2},$$

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- This gives a corner balance scheme where the value at the cell-center and at the cell edges are linear interpolations from the sub-cell values.

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- Static adaptivity using the spectral volume method has been shown to be successful in results from multidimensional computational fluid dynamics simulations (Wang, 2004).
- Furthermore, the interpolation inside each cell can be used to deal with dendritic meshes that arise in adaptive mesh refinement calculations.

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  - The spectral volume method is able to increase the number of Flops per communication by increasing K.
  - Of course, in discrete ordinates codes, increasing K means increasing the number of points where the scattering source must be stored (one for each sub-cell).

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  - This has been demonstrated in recent magnetohydrodynamics methods, possibly making the larger values of K "free".

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  - In the limit of optically thick, scattering dominated cells, the discretization limits to a discretization of the diffusion equation.
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$$\frac{\epsilon\mu_l}{\Delta x_{i,k}}\left(\hat{\psi}_l^{i,k+1/2}-\hat{\psi}_l^{i,k-1/2}\right)+\sigma_{\mathrm{t}}\psi_l^{i,k}=\frac{1}{2}\left(\sigma_{\mathrm{t}}-\epsilon^2\sigma_{\mathrm{a}}\right)\langle\psi_l^{i,k}\rangle+\frac{\epsilon^2 Q^{i,k}}{2}.$$

and expand ψ in a power series in ϵ

$$\psi_l^{i,k}=\sum_{j=0}^{\infty}\epsilon^j\psi_l^{(j),i,k},$$

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- We get an angular flux that is isotropic to leading order

$$\psi_l^{(0),i,k} = \frac{1}{2}\langle \psi_l^{(0),i,k} \rangle \equiv \frac{\phi^{(0),i,k}}{2}.$$

$$\frac{-2}{3\Delta x_i}\left[\frac{1}{\sigma_{\mathrm{t},i}\Delta x_i}\left(\phi^{(0),i,2} - \phi^{(0),i,1}\right) - \frac{1}{\sigma_{\mathrm{t},i-1}\Delta x_{i-1}}\left(\phi^{(0),i-1,2} - \phi^{(0),i-1,1}\right)\right] + \sigma_{\mathrm{a},i}\phi^{(0),i,1} + \sigma_{\mathrm{a},i}\phi^{(0),i-1,2} = Q^{i,1} + Q^{i-1,2}.$$

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- For the case of $K = 2$, if we turn the crank to get the diffusion equation, we get the consistent, diffusion discretization:

$$\frac{-2}{3\Delta x_i}\left[\frac{1}{\sigma_{\mathrm{t},i}\Delta x_i}\left(\phi^{(0),i,2} - \phi^{(0),i,1}\right) - \frac{1}{\sigma_{\mathrm{t},i-1}\Delta x_{i-1}}\left(\phi^{(0),i-1,2} - \phi^{(0),i-1,1}\right)\right] + \sigma_{\mathrm{a},i}\phi^{(0),i,1} + \sigma_{\mathrm{a},i}\phi^{(0),i-1,2} = Q^{i,1} + Q^{i-1,2}.$$

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  - Numerical results demonstrate the robustness of these methods.

Outline

Reed's Problem

- Reed's problem has several different material regions

Vacuum Bound.	Scattering Region $\Sigma_a = 0.1$ $\Sigma_s = 0.9$		Vacuum	Absorber $\Sigma_a = 5$ $\Sigma_s = 0$ $Q = 0$	Strong Source $\Sigma_a = 50$ $\Sigma_s = 0$ $Q = 50$	Reflect. Bound
	$Q = 0$ $x < 1$	$Q = 1$ $1 < x < 3$	$3 < x < 5$	$5 < x < 6$	$6 < x < 8$	

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  - $K = 2, 3, 4, 5, 6$ as well.
- We'll use S_8 and a sweep-based GMRES scheme to solve the SV equations.

Results with $K = 2, 3$

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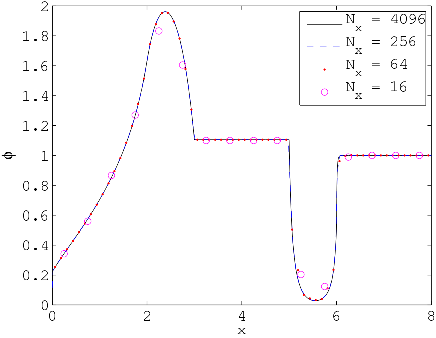
$K = 2$



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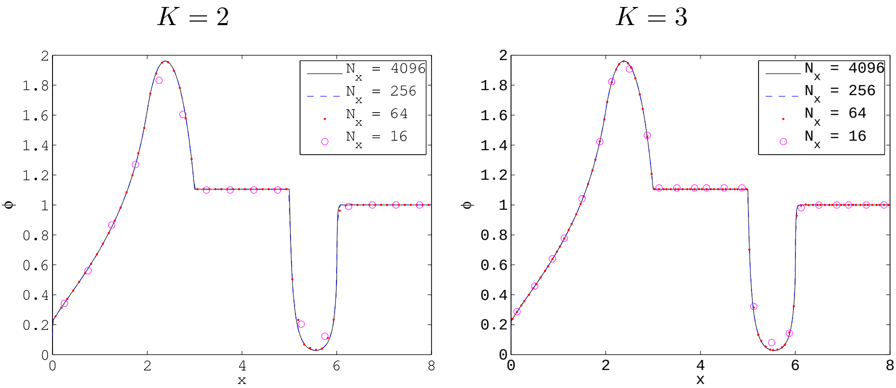

Results with $K = 4, 6$

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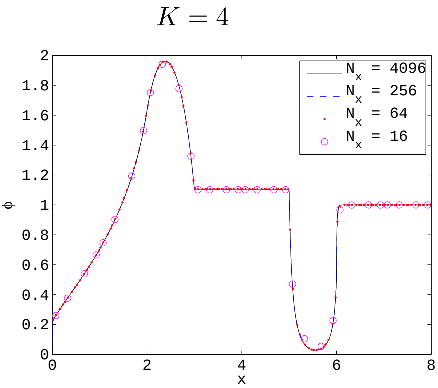

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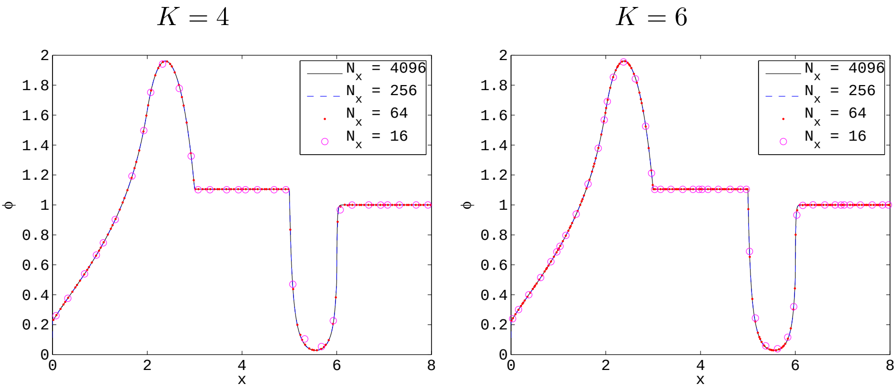

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- This problem is almost ideal for this method as the scalar flux is only non-smooth at the material interfaces.

Outline

Diffusive Problem (with boundary layer)

- This problem has a strong absorber next to a diffusive region

Isotropic Boundary	Absorbing Region	Strong Scattering Region	Vacuum Boundary
	$\Sigma_a = 2$ $\Sigma_s = 0$ $0 < x < 1$	$\Sigma_a = 0$ $\Sigma_s = 1000$ $1 < x < 2$	

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- For comparison we use a $K = 6$ and $N_x = 512$ solution as a reference ($\max \Delta x_{i,k} = .00097656$).

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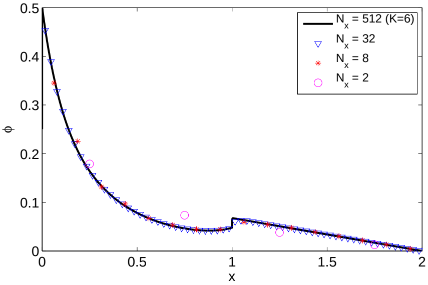

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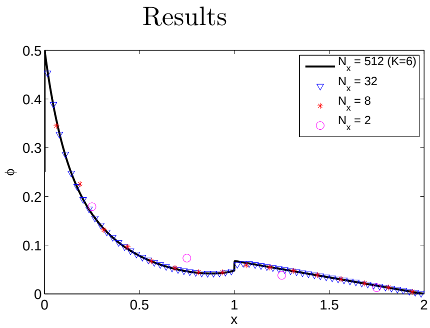

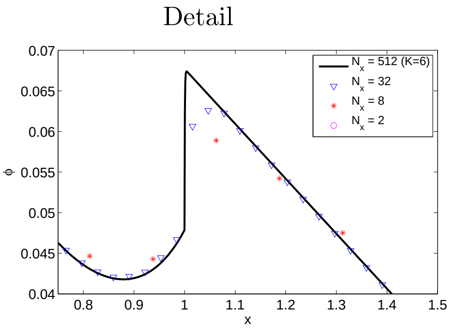

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- Away from the boundary layer the solution has the correct slope for $N_x = 8$ and 32.

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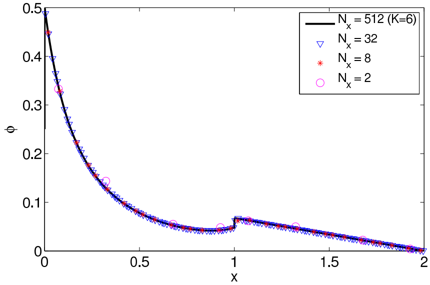

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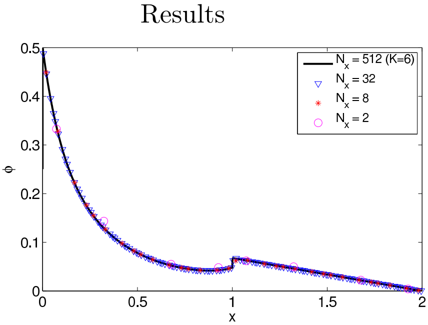

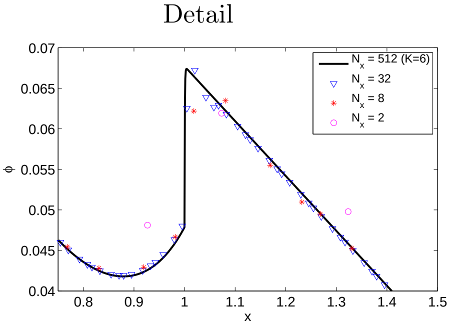

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    - These are due to interpolating across this sharp change.

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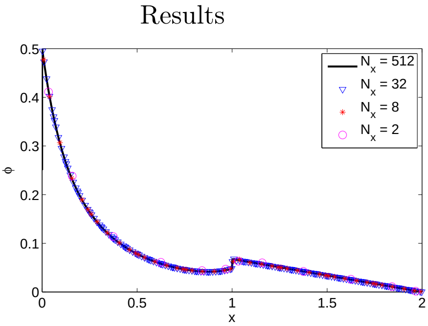

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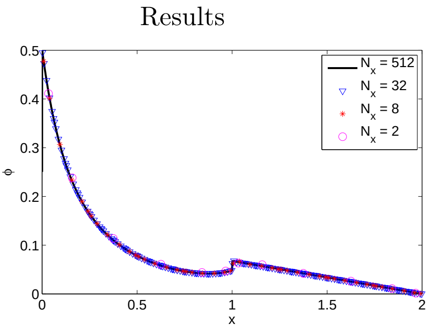

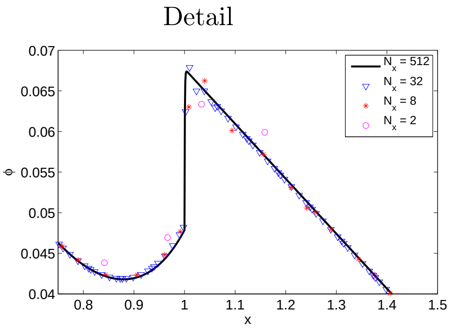

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- Might be a good candidate for local parallelism.

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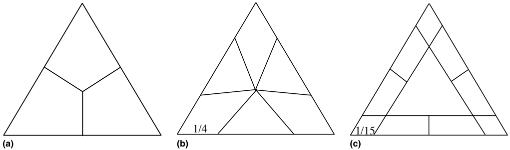
- This might necessitate fancier reconstruction schemes.

2-D Spectral Volume on Triangles

- Here's how Wang (2004) divided a triangle into subcells.



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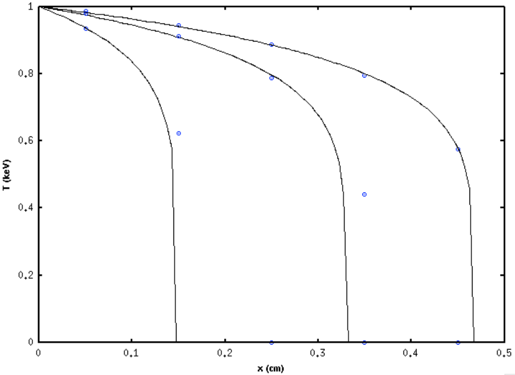
Marshak Wave

- This method also works for time-dependent thermal radiative transfer problems.



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- This method also works for time-dependent thermal radiative transfer problems.
- This figure shows the $K = 6$ solution with 5 cells and the analytic diffusion solution at $t = 10, 50$, and 100 ns for a problem with a 1 keV incident source and $\sigma = 300/T^3$.