The Asymptotic Drift-Diffusion Limit of Thermal Neutrons

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It is well known that in an infinite source, free, pure scattering medium with physically realizables cross-sections, the neutron distribution is a Maxwellian at the material temperature [1]. In this work we look at how small variations to these conditions change the behavior of the neutron distribution in space and energy. Specifically, our analysis will look at the influence of small amounts of scattering and a small source as well as temperature variations in the material. The result of the asymptotic analysis we perform shows that the neutron scalar flux satisfies a drift-diffusion equation where the diffusion coefficient and drift coefficients depend on the first-order scattering kernel as well as the variability of the material temperature. Below we give the highlights of this derivation.

We begin the derivation with the slab geometry transport equation,

$$\mu\frac{\partial\psi}{\partial x} + \left(\sigma_{\mathrm{s}}(E) + \sigma_{\mathrm{a}}(E)\right)\psi(x,\mu,E) = \int \mathrm{d}E' \int_{-1}^{1} \mathrm{d}\mu_0' \sigma_{\mathrm{s}}(E) f(E' \to E, \mu_0)\psi(\mu', E') + \frac{Q}{2} \tag{1}$$

where $\psi(x,\mu,E)$ is the angular flux, $\sigma_{\mathrm{s}}(E)$ is the scattering cross section, $\sigma_{\mathrm{a}}(E)$ is the absorption cross section, and Q is the source. The cross sections are quantities that depend on the local temperature. In the problem that we are interested in, we desire a large scattering cross section and a small absorption cross section along with a small source term. To do this we introduce the small parameter ϵ. We represent this scenario in the following manner: $\sigma_{\mathrm{s}}(E) \to \frac{1}{\epsilon}\sigma_{\mathrm{s}}(E)$; $Q \to \epsilon Q$; $\sigma_{\mathrm{a}}(E) \to \epsilon\sigma_{\mathrm{a}}(E)$. We substitute these definitions into (1) and obtain the following:

$$\mu\frac{\partial\psi}{\partial x} + \left(\frac{\sigma_{\mathrm{s}}(E)}{\epsilon} + \epsilon\sigma_{\mathrm{a}}(E)\right)\psi(x,\mu,E) = \int \mathrm{d}E' \int_{-1}^{1} \mathrm{d}\mu_0' \frac{\sigma_{\mathrm{s}}(E)}{\epsilon} f(E' \to E, \mu_0)\psi(\mu', E') + \epsilon\frac{Q}{2} \tag{2}$$

It can be shown that the leading order equations are

$$\sigma_{\mathrm{s}}(E)\psi^{(0)} = \int \mathrm{d}E' \int \mathrm{d}\mu' \sum_{\ell=0}^{\infty} P_\ell(\mu')\frac{2\ell+1}{2}\psi^{(0)}(\mu', E') f_\ell(E' \to E)\sigma_{\mathrm{s}}(E'), \tag{3}$$

where we have expanded ψ in a power series in ϵ

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \ldots \tag{4}$$

The solution to this infinite medium, pure scattering transport equation for any physically realizable cross-sections is the Maxwellian, i.e.,

$$\psi^{(0)} = \frac{\Phi(x)}{2}M(E,T), \tag{5}$$

with

$$M(E,T) \equiv \frac{E}{(kT)^2}e^{\frac{-E}{kT}}. \tag{6}$$

Moving on to the next order in ϵ we get a version of Fick's Law

$$J^{(1)} = -\frac{1}{3}[1 - \mathcal{S}_1]^{-1}\frac{d\phi^{(0)}}{dx}, \tag{7}$$

where the operator $\mathcal{S}_l$ is defined by

$$[\mathcal{S}_l]g(E) \equiv \frac{1}{\sigma_s(E)} \int \mathrm{d}E' \, \sigma_{sl}(E' \to E) g(E'). \tag{8}$$

It is possible to prove that the operator in Eq. (7) is invertible; we, however, will not do so in this summary. Then upon using Fick's law in the next order of the transport equation, one can obtain

$$-\frac{d}{dx} D(x) \frac{d}{dx} \Phi(x) + \frac{d}{dx} b(x) \Phi(x) + \bar{\sigma}_{\mathrm{a}} \Phi(x) = \bar{Q}, \tag{9}$$

where

$$D(x) = \frac{1}{3} \left[\int_0^\infty \mathrm{d}E \, [1 - \mathcal{S}_1]^{-1} M(E, T) \right], \tag{10}$$

and

$$b(x) = -\frac{1}{3} \frac{dT}{dx} \left[\int_0^\infty \mathrm{d}E \, [1 - \mathcal{S}_1]^{-1} \frac{\partial M}{\partial T} \right]. \tag{11}$$

This is a drift diffusion equation that governs the relationship between the magnitude of the energy-integrated scalar flux and the spatial variation of the material temperature. In many respects this is similar to the equilibrium-diffusion limit of the radiative transfer equations in that the equilibrium distribution (in that case a Planckian) satisfies a diffusion equation that is dependent on the underlying material temperature.

References

[1] George I. Bell and Samuel Glasstone. *Nuclear Reactor Theory*. Robert E. Kreiger Publishing, Malabar, Florida, 1970.