## A Flux-Limited Diffusion Method for Simulating Radiative Shocks

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## INTRODUCTION

As demonstrated in recent work [1, 2], in some radiative shock profiles the presence of an optically thin cooling layer behind the density jump can lead to radiation and material temperature profiles that are qualitatively inaccurate when modeled using a diffusion treatment. This previous work, and other published solutions [3, 4], only considered diffusion models based on Fick's law with a constant Eddington factor. The remaining open question is whether flux-limited diffusion treatments could perhaps ameliorate the deficiencies of simple diffusion treatments. An initial simulation study using the Larsen flux limiter [5], shown in Fig. 1, seemed to indicate no such fix was in the offing using standard flux-limiters. Nevertheless, studying Fig. 1 suggests that the Larsen flux limiter with $n < 1$ could give the correct qualitative behavior (in the figure as $n$ increases the flux-limited diffusion solution looks less like the transport solution). Having $n < 1$ makes the flux limiter "turn-on" more readily. Below we develop a new flux limiter that turns on specifically in the cooling layer of a radiative shock. Our calculations in this work were for radiation-only problems where the temperature profile is chosen to match that of a shock profile. Future work will explore the impact of these flux-limiters on radiation-hydrodynamics problems.

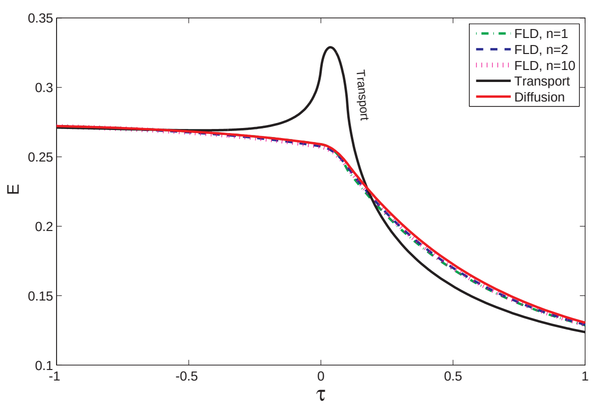



Fig. 1: Energy density solutions using various integer $n$ values when applied to a temperature profile with a thin cooling layer.

For simplicity, the gray transport approximation is used and we neglect scattering. The plasma is modeled by one temperature (i.e., assuming electrons and ions are in thermal equilibrium). Under these conditions, the steady radiative transfer equation is,

$$\Omega \cdot \nabla\psi = \sigma_a \left(\frac{ac}{4\pi}T^4 - \psi\right), \tag{1}$$

where $T$ is the material temperature, $\Omega$ is the angular or directional variable, $\sigma_a$ (length$^{-1}$) is the macroscopic absorption cross section, $c$ (distance/time) is the speed of light, and $a$ (energy/temperature$^4$-volume) is the radiation constant. $\psi = \psi(\Omega, x)$ and is the radiation intensity (energy/area-time-steradian). A common simplification of Eq. (1) is the diffusion model given by [6]

$$-\frac{d}{dx}D\frac{d}{dx}E + \sigma E = \frac{a}{4\pi}T^4, \tag{2}$$

where $D$ is a diffusion-coefficient and $E = \frac{1}{c}\int_{4\pi}\psi\, d\Omega$. For the flux-limited diffusion (FLD) method, the Larsen flux-limiter,

$$D = \left[(3\sigma)^n + \left(\frac{|\nabla E|}{E}\right)^n\right]^{-1/n}, \tag{3}$$

was used. The salient feature of this flux-limiter is that when the energy density is relatively constant, as is the case further away from the shock front, the diffusion coefficient returns to the Fick's law prescription: $D = \frac{1}{3\sigma}$.

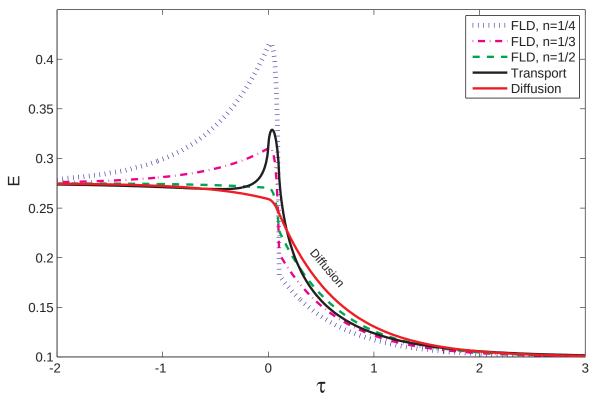
In this work we solve the general problem where the material has a temperature profile given by

$$acT^4(\tau) = \begin{cases} \alpha & \tau < 0 \\ 1 & \tau \in [0, \tau_0] \\ \beta & \tau > \tau_0 \end{cases},$$

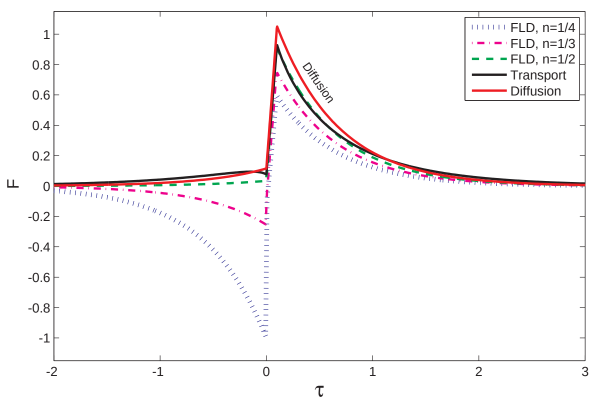
where we have written our spatial variable in terms of mean-free paths: $\tau = \sigma x$. This problem setup is a simplified version of a steady radiative shock profile; the region beyond $\tau_0$ is the precursor region when $\beta < \alpha$, and the region between 0 and $\tau_0$ is the cooling layer, and the region of negative $\tau$ is the cooling layer with $a < 1$. In Figs. 1- 3 we have $\alpha = 0.275$, $\beta = 0.1$, and $\tau_0 = 0.1$, so that the cooling layer is a tenth of a mean-free path. Figure 4 has a thicker cooling layer and less of a difference in emission source between the regions: $\alpha = .8537$, $\beta = .7807$, and $\tau_0 = 0.5$.

## NEW METHOD

Flux-limited diffusion codes commonly use $n = 2$, but this will not correctly reproduce the spike in energy density directly downstream of the shock front. This is commonly referred to a a Zel'dovich spike and is present in many physically realizable systems. After noting that larger values of $n$ did not produce a spike it was realized (through the clarity of hindsight) that in a diffusive model the flux must become negative (radiation moving from right to left) in a region for the spike to be produced.




(a) Energy Density




(b) Radiation Flux

Fig. 2: Energy density and radiation flux for various fractional $n$ values applied to a problem with a thin cooling layer.

Only the transport solution and fractional values of $n$ yield a spike. Although using a fractional value of $n$ throughout all regions will yield an inaccurate shock profile according to numerical experiments, diffusion solutions with fractional values of $n$ do capture the transport behavior within the cooling layer adequately. To resolve these inconsistencies, a fractional value of $n$ was used only if $T \geq T_{\max}$. Anywhere this relationship was not satisfied, the common $n$ value of 2 was used. This method produces shock profiles that approximate the transport solutions of these problems accurately throughout each region.

Looking at different values of $n$ for the cooling layer in Fig. 2 suggests that $n = 1/3$ is the best value of those tried. Also in this figure we see that the radiation flux becomes negative in the cooling layer. However, in the transport solution it is positive throughout the problem. In some sense we have traded accuracy in the radiation flux to have a more accurate energy density through the shock profile: for a diffusion model it is not possible to have a positive radiation flux throughout the profile *and* have a spike in the energy density. The value of $1/3$ for $n$ seems to be an acceptable value for the flux-limiter over a range of problems, and future theoretical work will have to be performed to understand the optimal way to pick this value as well as possible means to automatically vary it as the simulation progresses.

A crucial factor in the accuracy of flux-limited diffusion is the thickness of the cooling layer. If the cooling layer is relatively thick, in terms of mean-free paths, then the diffusion model can react to the spike in the emission source and accurately reproduce a “bump". This can be seen in Fig. 4. Although the diffusion solution does not exactly reproduce the transport solution’s “bump", it is qualitatively showing the correct form. Meanwhile, as shown in Fig. 3, a thin cooling layer leads to the diffusion solution being qualitatively incorrect. Within the cooling layer the transport solution shows an abrupt spike whereas the diffusion solution is monotonically decreasing.

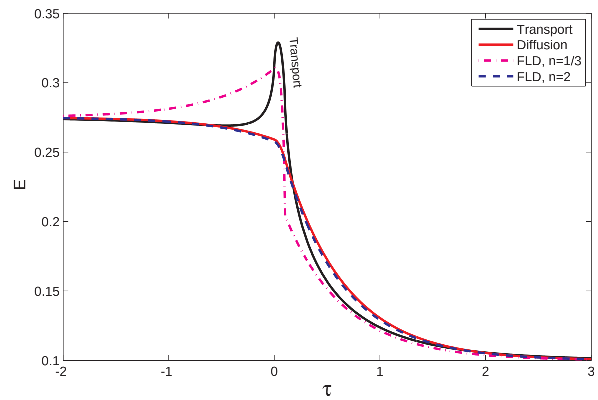



Fig. 3: Transport and diffusion solutions for a thin cooling layer.

The selection of $T_{\max}$ is not arbitrary and can be readily selected for a radiative shock simulation: from the maximum principle [7] we know that for a pure radiative transfer problem in the absence of sources the temperature must remain less than or equal to the maximum temperature of the initial and boundary conditions. Therefore, in a radiative shock simulation, $T_{\max}$ would be the maximum temperature of the initial and boundary conditions because any temperatures in the solution higher than this value would necessarily be due to shock heating (exactly the type of heating phenomenon we are trying to accurately capture).

## CONCLUSION

Our new prescription for flux-limited diffusion has been shown to more accurately approximate transport solutions for problems with either a thin or thick cooling layer, as shown in Figs. 3 and 4. When shocks have a spike in energy density directly downstream from the shock front, it is advantageous to use the fractional value $n = 1/3$ inside the cooling layer. This value yields a spike of approximately the same magnitude, and matches the transport solution outside of the adaptation zones that are immediately adjacent to the cooling layer. It overestimates the energy density immediately after the cooling

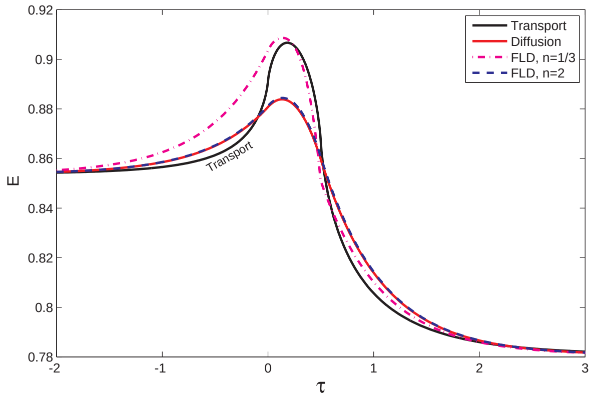



Fig. 4: Transport and diffusion solutions for a thick cooling layer.

layer, and underestimates the energy density right before the shock front. For $n = 2$ it is shown that this properly matches the diffusion solutions for problems with either thick or thin cooling layers. The "turn-on" feature can be implemented into shock simulation codes to accurately solve for energy density and flux profiles, using $n = 1/3$ if $T \geq T_{\max}$ and $n = 2$ elsewhere. For future work we plan on implementing this method into radiation hydrodynamics simulations to evaluate the impact of the flux-limiter on radiative shock simulations.

## ACKNOWLEDGEMENT

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