Self-Similar Radiation-Hydrodynamics Solutions in the Equilibrium Diffusion Limit

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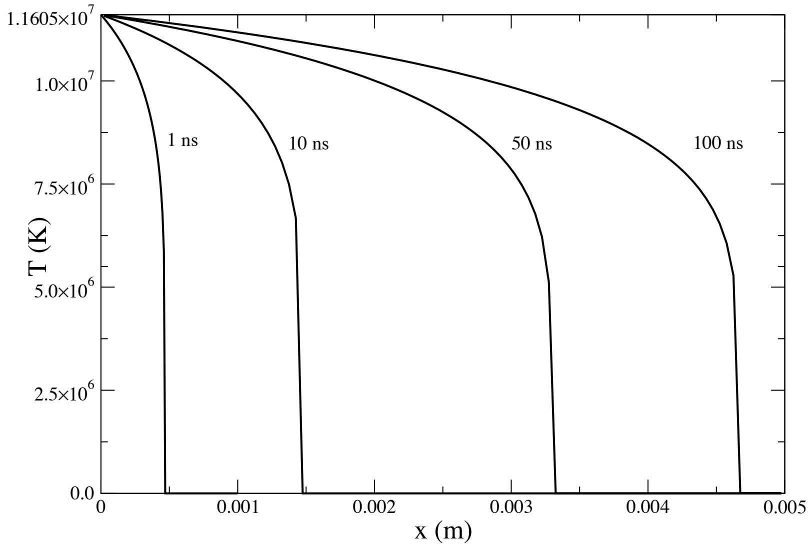
The Classic Marshak Wave

- The Marshak wave is a special, soluble problem in radiation hydrodynamics and can arise in high-energy-density experiments.
- The problem has radiation striking a cold material. As the material heats a wave of thermal energy propagates into the material.
- An example of this type of wave could be radiation emitted from a hohlraum striking a fusion target.
- The classical version of this problem is valid in the regime where the radiation energy flux is high, but the radiation energy density is negligible in regard to the material internal energy.
- Also, the material is considered to be stationary:
  - Radiation energy impinges on a quiescent material
  - Drives a radiation wave that travels faster than the speed of sound in the material.

Marshak Wave



Previous Work

- The eponymous, original solution of R.E. Marshak comes from 1958 (in the volume 1, number 1 of *Physics of Fluids*)
- Petschek et al., gave a very nice description of how to obtain the solutions, including allowing temperature dependent opacities and densities in a 1960 report (LAMS-2421).
- Every monograph on HEDP or related fields includes a discussion of the Marshak wave (Drake, Castor, Z&R).
- In a 2009 LANL technical report, Nelson and Reynolds delve into gory details of how to obtain Marshak wave solutions using `Mathematica`.
- It is also worth pointing out that the theory of admissible self-similar solutions in covered in some detail by Coggeshall and Axford.

Code Verification

- Recently, there have been several new studies of rad-hydro phenomenon
  - Lowrie and co-authors, and more recently Ferguson and Morel, have published several studies of radiating shock waves using different radiation models.
  - There has also been developments in the theory of such shocks in different regimes.
- Part of the motivation for these recent papers on radiation hydrodynamics behavior has been the necessity of verifying numerical simulation codes for radiation hydrodynamics.
- Verification in this sense means demonstrating that the code is solving the intended equations and that numerical errors behave as expected (e.g., going to zero at the correct rate as the mesh is refined).

New Solutions Mean Better Code Verification

- The classic Marshak wave, typically only tests the ”radiation” part of a radiation hydrodynamics code.
- It cannot test that radiation coupling terms (e.g., momentum deposition) are properly accounted for
  - Except that when the velocity is zero, the solution behaves as expected.
- To that end our new solutions introduce material motion to the Marshak wave problem.
- Our solutions can be used in code verification, regardless of the radiation transport model employed in the code (e.g., flux-limited diffusion, discrete ordinates, Monte Carlo, etc.)
  - Because the solution is given in the equilibrium diffusion limit.
  - A limit that all reasonable implementations of a radiation package posses.

Hydrodynamics Model

The Euler equations, in non-dimensional form are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0\,,$$

$$\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho \vec{v} \otimes \vec{v}) + \nabla p = -\mathbb{P} S_{\mathrm{F}}\,,$$

$$\frac{\partial}{\partial t}(\rho E) + \nabla \cdot [(\rho E + p)\,\vec{v}] = -\mathbb{P}\mathbb{C} S_{\mathrm{E}}\,.$$

E, the total specific energy is given by

$$E = e + \frac{1}{2} v^2, \tag{1}$$

where $v = |\vec{v}|$ and e is the internal specific energy with a simple equation of state $e = c_{\mathrm{v}} T$.

The reference sound speed, a_∞, is given using the specific heat c_{v} as

$$a_\infty = \sqrt{c_{\mathrm{v}} T_\infty}.$$

Non-Dimensional Parameters

- In our model we have included the non-dimensional parameters

$$\mathbb{C} = \frac{c}{a_\infty} = \frac{c}{\sqrt{c_\mathrm{v} T_\infty}}, \qquad \mathbb{P} = \frac{a_\mathrm{r} T_\infty^4}{\rho_\infty a_\infty^2} = \frac{a_\mathrm{r} T_\infty^3}{\rho_\infty c_\mathrm{v}}, \tag{2}$$

  where c is the speed of light and $a_\mathrm{r} = 4\sigma_\mathrm{SB}/c$ is the radiation constant with σ_SB the Stefan-Boltzmann constant.
- The interpretation of these parameters is as follows:
  - $\mathbb{C}$ is a measure of how relativistic the flow is, and
  - $\mathbb{P}$ is a measure of how much energy is in the radiation field compared with the material internal energy.
- When $\mathbb{P}$ is small but $\mathbb{PC}$ is order 1, this is the flux-dominated regime: There is no momentum coupling, but the flux of radiative energy affects the total energy equation.
- When $\mathbb{P}$ is non-negligible this is the radiation energy-dominated regime where momentum coupling is important.

Radiation Model

For the radiation transport we will employ a P_1 model

$$\frac{\partial E_{\mathrm{r}}}{\partial t} + \mathbb{C}\nabla\cdot F_{\mathrm{r}} = \mathbb{C}S_{\mathrm{E}}, \qquad \frac{\partial F_{\mathrm{r}}}{\partial t} + \frac{1}{3}\mathbb{C}\nabla E_{\mathrm{r}} = \mathbb{C}S_{\mathrm{F}},$$

where

$$S_{\mathrm{E}} = \sigma(T^4 - E_{\mathrm{r}}) + \sigma\frac{\vec{v}}{\mathbb{C}}\cdot F_{\mathrm{r0}}, \qquad S_{\mathrm{F}} = -\sigma F_{\mathrm{r0}} + \sigma\frac{\vec{v}}{\mathbb{C}}(T^4 - E_{\mathrm{r}}),$$

$$F_{\mathrm{r0}} = F_{\mathrm{r}} - (\vec{v}E_{\mathrm{r}} + \frac{\vec{v}}{3}E_{\mathrm{r}})/\mathbb{C}.$$

- E_{r} is known as the radiation energy density and it is proportional to the zeroth angular moment of the radiation specific intensity.
- F_{r} is known as the radiation flux and it is proportional to the first angular moment of the radiation specific intensity.
- The material absorption opacity (with units of inverse length) is given by σ.

Operator Splitting

- In typical applications the RHD model (i.e., the coupled Euler and radiation equations) are solved in an operator split fashion,
  - where the P_1 equations (or some other transport model) is solved
  - coupled with a material internal energy equation that contains the radiation-matter coupling terms only:

$$\frac{\partial \rho e}{\partial t} = -\mathbb{P}\mathbb{C}\boldsymbol{S}_{\mathrm{E}}.$$

- The other terms in the total energy equation are updated during the hydrodynamics solve, along with a correction to take into account momentum exchange.
- As part of the operating splitting procedure, the radiation solve is undertaken with the density and velocity terms evaluated at a particular time level.

The equilibrium diffusion limit

- If we take the radiation solve part of the model (i.e., radiation model plus part of the internal energy equation), and make the following scaling:
- The absorption cross-section is very large,

$$\sigma \to \frac{\sigma}{\epsilon},$$

- The ratio of the speed of light to the speed of sound is also very large,

$$\mathbb{C} \to \frac{\mathbb{C}}{\epsilon}.$$

- We get, to leading order, a nonlinear advection-diffusion equation that is solved during the radiation step:

$$\frac{\partial}{\partial t}\left(\rho c_{\mathrm{v}} T + \mathbb{P} T^4\right) + \mathbb{P}\frac{4}{3}\nabla \cdot \vec{v}\, T^4 = \nabla \cdot \frac{\mathbb{C}\mathbb{P}}{3\sigma}\nabla T^4.$$

Relationship to Classical Marshak Wave Problem

- The full-blown equation, in slab geometry is

$$\frac{\partial}{\partial t}\left(\rho c_{\mathrm{v}} T + \mathbb{P} T^4\right) + \mathbb{P}\frac{4}{3}\frac{\partial}{\partial x} v T^4 = \frac{\partial}{\partial x}\frac{\mathbb{CP}}{3\sigma}\frac{\partial}{\partial x} T^4.$$

- The classical Marshak wave solves this equation in the limit of $\mathbb{P} \to 0$ and $\mathbb{PC} \to 1$:

$$\frac{\partial}{\partial t}\rho c_{\mathrm{v}} T = \frac{\partial}{\partial x}\frac{1}{3\sigma}\frac{\partial}{\partial x} T^4.$$

- Notice how the advection term naturally drops out (without assuming $v \to 0$) and the time derivative term simplifies.

Problem Definition

- We want to solve

$$\frac{\partial}{\partial t}\left(\rho c_{\mathrm{v}} T + \mathbb{P} T^4\right) + \mathbb{P}\frac{4}{3}\frac{\partial}{\partial x} v T^4 = \frac{\partial}{\partial x}\frac{\mathbb{C}\mathbb{P}}{3\sigma}\frac{\partial}{\partial x} T^4.$$

- The problem we will solve has an initially cold, semi-infinite material located at $x \geq 0$.
- At time $t = 0$ a radiation source is turned at $x = 0$,
- The radiation source is a blackbody at $T = 1$.
- We also allow a temperature dependent opacity

$$\sigma = \kappa_0 T^{-n}.$$

Self-Similar Scaling

- As is typical for diffusion problems we define a self-similar variable
$$\xi = \frac{Ax}{\sqrt{t}}.$$
- Without the advection term, this is all you need to do.
- To make the advection term amenable to a self-similar solution profile we prescribe
$$v(t) = \frac{\theta U}{\sqrt{t}}.$$
- We do not consider how such a form for the velocity might be formed,
  - Indeed, in a simulation code such a dependence could be prescribed.
- Also, the velocity is uniform in space.

Self-Similar Problem Definition

- Upon inserting our scaling we get, after some interesting algebra and setting A and U for convenience,

$$-\xi \frac{d}{d\xi}\left(T + \mathbb{P}T^4\right) + \mathbb{P}\theta \frac{d}{d\xi} T^4 = \frac{d^2}{d\xi^2} T^{(n+4)},$$

- The problem definition has also changed.
  1. At $\xi = 0$, $T = 1$.
  2. The temperature in front of the wave is cold (i.e., $T = 0$).
- We refer to the value of ξ beyond which $T = 0$ as ξ_{max}.
- For a particular value of ξ_{max}, there are an infinite number of solutions that tend to zero.
  - Only one of these solutions maintains a zero radiation flux in the limit $\xi \to \xi_{max}$.
- Also, only one ξ_{max} will match the boundary condition at $\xi = 0$.

Solution Procedure

- Castor outlines a numerical procedure where one
  1. Guesses a ξ_{max},
  2. Integrates the ODE backwards to $\xi = 0$,
  3. Adjusts ξ_{max} based on $T(0)$, (e.g. if $T(0) > 1$ then decrease ξ_{max})
  4. Repeats 2-3 until convergence.
- The difficult part is coming up with an approximate value of T just behind ξ_{max} to start the integration.
- The rest of the procedure can be handled using a numerical ODE integrator and root finder.
- Nelson and Reynolds describe in detail how to accomplish this for the classical Marshak wave using `Mathematica`.

Approximate T near the wavefront

- By integrating the ODE from ξ to ξ_{max} and making a series of approximations, one gets that near the wavefront

$$T(\xi) = \left[\frac{n+3}{n+4} \left(\xi_{max} - \xi \right) \left(\xi_{max} + \mathbb{P} \frac{n+3}{n+6} \left(\xi_{max} - \theta \right) \left(\frac{n+3}{n+4} \xi_{max} \left(\xi_{max} - \xi \right) \right)^{3/(n+3)} \right) \right]^{1/(n+3)} .$$

- We use this value to step away from ξ_{max}.
- That is, the initial condition for the ODE integration is $T(\xi_{max} - \delta)$ using the formula above.

Solutions with $v = 0$

- If we set $v = 0$, in our scaling $\theta = 0$, we can see the effect of non-negligible $\mathbb{P}$ on the solution.
- We will look at two common cases for the behavior of $\sigma = \kappa_0 T^{-n}$, $n = 0$ and $n = 3$.
- The values for ξ_{max} obtained by Nelson and Reynolds are
  - $\xi_{max} = 1.23117$ in the $n = 0$ case.
  - $\xi_{max} = 1.11993$ in the $n = 3$ case.
- Based on solutions using several values of $\mathbb{P}$ from 0 to 2 we found that including radiation energy in the wave that the wave slows down.
- A least-squares fit of ξ_{max} as a function of $\mathbb{P}$ gives:

$$\xi_{max} \approx 0.032\mathbb{P}^2 - 0.19\mathbb{P} + 1.2 \qquad n = 0,$$

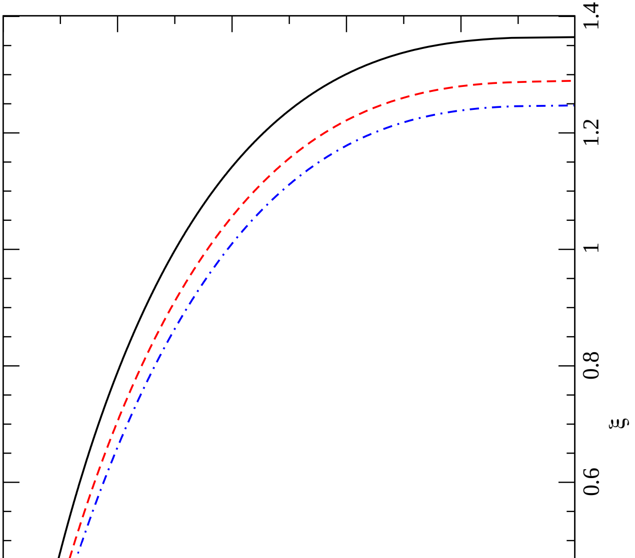
$$\xi_{max} \approx 0.046\mathbb{P}^2 - 0.25\mathbb{P} + 1.1 \qquad n = 3.$$

Solutions with advection

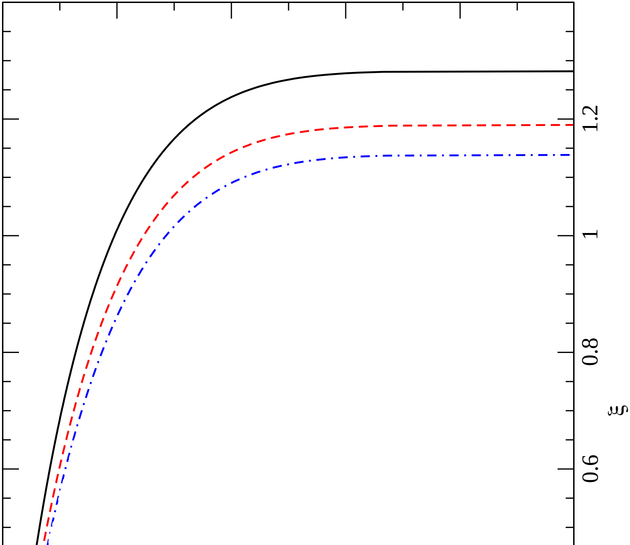
Table: ξ_{max} for Motion Problems

$\mathbb{P}$	θ	ξ_{max}	
		$n = 0$	$n = 3$
0.04573	10	1.36449	1.28213
	5	1.28845	1.18959
	2	1.24712	1.13852
	1	1.23399	1.12220
	0.1	1.22243	1.10780
	0	1.22116	1.10622
0.5	10	3.57574	3.52816
	5	2.07107	2.01057
	2	1.40900	1.31516
	1	1.25937	1.14168
	0.1	1.14910	1.00925
	0	1.13806	0.99582
1	10	5.19830	5.16451
	5	2.83272	2.76374
	2	1.57627	1.47967
	1	1.28225	1.15790
	0.1	1.08784	0.93103
	0	1.06966	0.90932

Solutions with advection, $n = 0$, $\mathbb{P} = 0.04573$



Solutions with advection, $n = 3$, $\mathbb{P} = 0.04573$



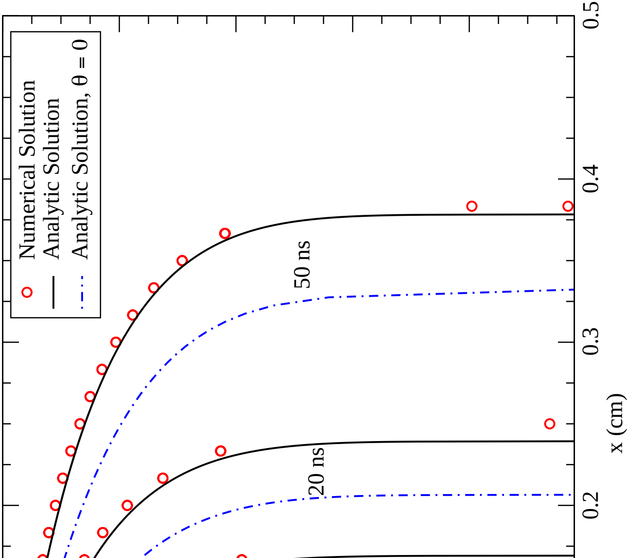
Solutions as code verification

- We can use our solutions to test the radiation package of a radiation-hydrodynamics code.
- Run the code where the hydro solver always tells the radiation package that $v(t) = \theta U / \sqrt{t}$.
- This will test the v/c coupling terms in the radiation solve with no underlying hydrodynamics error.
- In principle there could be an issue with evaluating $u(0)$ because of the singularity.
- In practice most radiation solvers expect to receive the velocity evaluated at either the end of the time step or some intermediate time level.

Verification Test

- To demonstrate how this can be done we took a "typical" Marshak wave problem from the literature:
- $c_\mathrm{v} = 0.1\,\mathrm{GJ/g\,keV}$, $\rho_\infty = 3.0\,\mathrm{g/cm^3}$, $\kappa_0 = 300$, $T_\infty = 1.0$ keV, $l = 1$ cm, and $n = 3$.
- Using these parameters, $\mathbb{C} = 948.027$ and $\mathbb{P} = 0.04573$.
- We then ran the problem using an existing transport code with the velocity evaluated at the middle of each time step.
- Below we show the solution for 3 different times: 10, 20, and 50 nanoseconds.
- Not a full verification study of the code presented here, but does show the solutions are obtainable "out-of-the-box" without much code modification.

Comparison with Numerical Solutions



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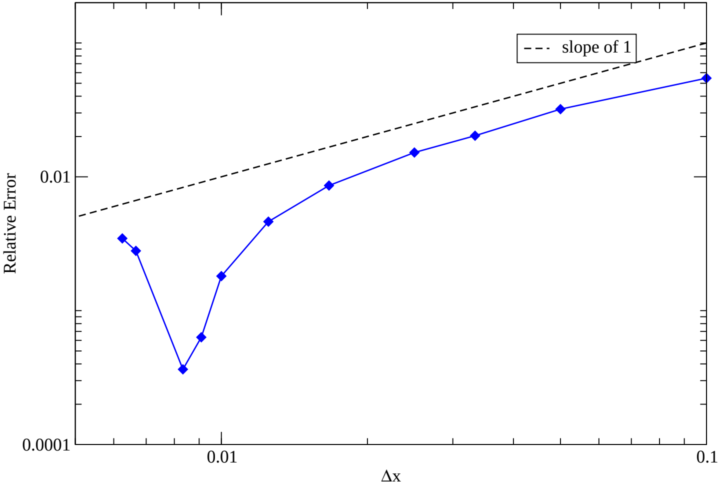

Summary and Conclusions

- We have presented a new radiation-hydrodynamics self-similar solution.
- Extended the classical Marshak wave to have non-negligible radiation energy density and material motion.
- We have shown that these solutions can be used in novel code verification exercises.

Possible future directions:

- We could do this in different geometries (e.g. cylindrical or spherical)
- It might be possible to do a problem without a driving source, rather with an delta-function of energy at $x = 0$ and $t = 0$.